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THE CORPUSCLE - A SIMPLE BUILDING BLOCK FOR POLYHEDRAL NETWORKS

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ABSTRACT: The corpuscle is a geometric structure formed by ten regular, but slightly deformable triangles. It allows for building a variety of three-dimensional structures, including the Goldberg icosahedron, infinite chains, closed rings consisting of 8, 12 or 16 subunits, and a cube-like structure comprising 20 subunits. These shapes can be built as paper models, which allow for a slight deformation of the triangles. In general, such deformations are necessary to obtain rings and other closed structures. Some of the structures are flexible, i.e. their geometric shape can be varied continuously with no or little distortion of the surface triangles. We distinguish between mathematical flexibility (where we require exactly regular triangle faces) and approximate, physical deformability (where edge lengths can be slightly distorted). Flexibility appears, for instance, in corpuscle chains: movements of a single element lead to collective conformations change along the chain, the so-called breathing. The resulting conformations of individual subunits repeat each other, approximately, after three units. Owing to this approximate periodicity, rings built from 8 or 16 corpuscles are rigid, while the 12-ring can be deformed quite easily, which is confirmed by the paper models. Three-dimensional corpuscle networks can be constructed by arranging regular or truncated octahedra in periodic patterns and decorating them with corpuscle balls.

Keywords: Corpuscle, Goldberg icosahedron, regular triangle, multi-stable polyhedron, periodic structure, flexible geometric body

1. INTRODUCTION

Three of the five Platonic solids - the tetrahedron, the octahedron, and the icosahedron - are bounded by regular triangles, but such triangles can also form various other, less symmetrical structures. A variety of polyhedra can be constructed from a simple building block, the so-called *corpuscle*. Its basic shape is a pentagonal double pyramid (see Figure 1, left). By cutting the edges between two of its five segments, we obtain an open side (a "mouth") that allows for plugging in another corpuscle unit. This construction can be further extended: corpuscles with two mouths give rise to straight or ring-shaped chains, while corpuscles with three mouths allow for building branched networks. All these structures are bounded by regular - or possibly slightly distorted - triangles. In this paper, we shall describe a number of such structures: we start with the Goldberg icosahedron, a structure formed by two connected corpuscle subunits, and construct, eventually, periodic networks of corpuscle balls, each comprising twenty corpuscle subunits in a symmetric arrangement.

Some of the structures can show collective motions, which leave all edge lengths and the shapes of all surface triangles unchanged. We call such structures *flexible*. If motions require a stretching of the edges, the structure is called *deformable*. In both cases, the different shapes of a structure are termed *conformations*. Slight deformations may also be needed to obtain closed, ring-like structures.

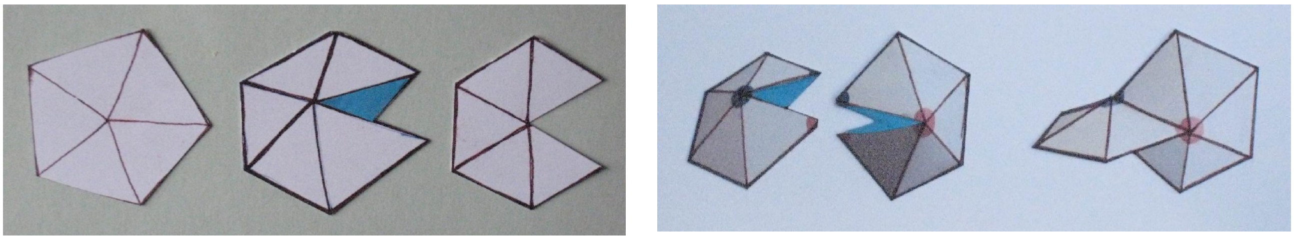


Figure 1: Corpuscles and the connection between them. Left: a corpuscle in three possible conformations (from left to right). In the closed conformation, the corpuscle forms a pentagonal double pyramid (seen from top); by a cut between two segments, a mouth arises. It becomes wider as the corpuscle's central axis is contracted. As the mouth reaches its maximal width, the axis shrinks to a point and the corpuscle is flat. Right: the Goldberg icosahedron can be built by connecting two open corpuscles mouth to mouth. Red and blue dots determine a unique orientation for the connection (see text).

While mathematical proofs about the existence and flexibility of corpuscle structures can be tough, some insights may be obtained from paper models. If not stated otherwise, we shall consider here corpuscles with deformable edges. Further information about corpuscle structures can be found at the web site www.korpuskel.de.

2. CORPUSCLES AND CONNECTIONS BETWEEN THEM

2.1 The corpuscle

The corpuscle in its basic form consists of ten regular triangles in the following arrangement: five triangles form a pentagonal pyramid. The other five yield another pyramid and both pyramids are joined bottom to bottom. A face on top and its counterpart on the bottom form a *segment*. The edges between triangles are thought to be flexible, so the angle between adjacent triangles can change. As long as all ten triangles are connected, the overall structure is rigid. However, if we cut the corpuscle along the edges between two of the segments, a mouth appears and the entire structure becomes flexible: we can deform it from the original double pyramid down to a double layered, flat hexagon in which one of the triangles is missing (see Figure 1, left).

These different conformations can be described by a single conformation parameter, the thickness b (length of the corpuscle's central axis), which then determines the mouth width a (compare Fig. 5). If we consider triangles of edge length 1, simple trigonometry shows that $a = 2\, r \sin(\pi - 5 \arcsin(1/(2\, r)))$, where $r=\sqrt{1-(b/2)^2}$ is the radius of the circle through the six vertices in the corpuscle's central plane and arcsin is the inverse of the sine function. The value of b can vary between

$$b=2\sqrt{1-(2\sin(\pi/5))^{-2}}\approx 1.0515$$

(thick conformation) and b=0 (flat conformation); at the same time, a varies between a=0 and a=1.

Figure 2: Deformation of the Goldberg icosahedron (Fig 1, right). Left: in one extreme position: the right unit is completely flat. Centre: in the neutral position, both units have the same shape. Right: in the second extreme position, the left unit is completely flat.

2.2 The Goldberg icosahedron

If we connect two corpuscles mouth to mouth, we obtain the *Goldberg icosahedron* [1] shown in Fig. 1, right. Just like the Platonic icosahedron, it is bounded by twenty regular triangles. The central axis of one corpuscle connects the tips of the other unit that arise from the cut - and *vice versa*.

(a) (b)

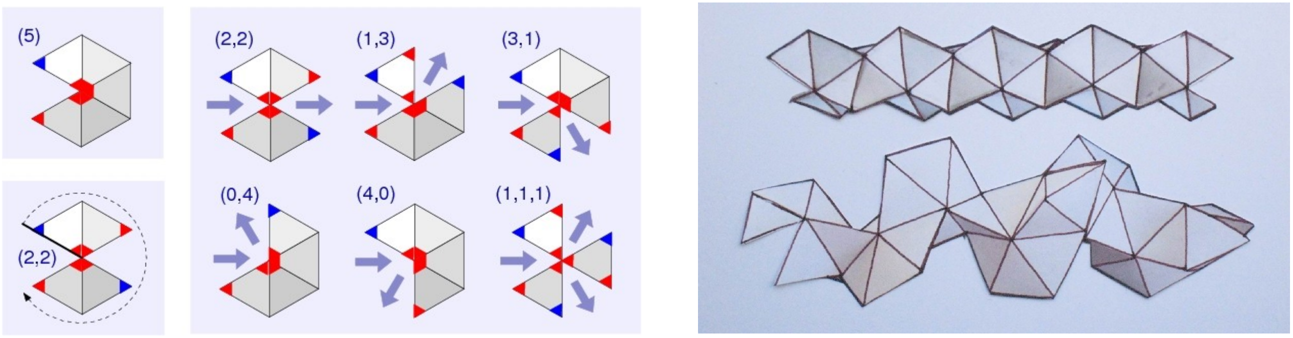


Figure 3: Corpuscles as building blocks (a) Top left: basic corpuscle in flat conformation, seen from top (see Fig. 1). Bottom left: to build chains, a second mouth is needed. The new mouth replaces one of the existing segments. Our short notation works as follows: we look from the top (red dot in the centre), start at the thick line, and describe the groups of connected segments in clockwise direction order. In this case, we obtain two segments, a mouth, and two more segments, or briefly, (2,2). Right: all five corpuscle types with four segments and two mouths and the branch point type (1,1,1) with three segments and three mouths. Corpuscles are attached to an existing chain with their left mouth (incoming arrow). Outgoing arrows show directions in which the chain can be extended. (b) Corpuscle chains. Top: straight type (2,2) units form a straight chain consisting of alternating vertical and horizontal units. Bottom: type (3,1) units give rise to a helical chain.

The conformations of the two units are coupled: if one unit has a thick shape (long central axis, narrow mouth), the other one is flat (short axis, wide mouth).
A Goldberg icosahedron with regular triangle faces can display three distinct conformations [1]: in the neutral conformation, both corpuscle units show the same shape. Besides this, there are two asymmetrical conformations in which one of the units is almost flat, while the other one is thick. A continuous movement between these conformations requires an elastic deformation of the edge lengths. In a simple mechanical model, we can represent the edges by elastic springs; any deviation from the standard edge length 1 results in a force. In this model, only the three above-mentioned conformations display a mechanic equilibrium in which no forces act on the structure. Therefore, Goldberg called his icosahedron a *multi-stable polyhedron*.
The corpuscle's volume depends on its conformation and vanishes if it is completely flattened. As a consequence, the Goldberg icosahedron changes its volume during its motion. In paper models, this can be experienced as an audible breathing as air is blown in and out.

3. CORPUSCLES WITH TWO MOUTHS CAN FORM CHAIN STRUCTURES

3.1 Corpuscle chains

If we remove one more segment from the basic corpuscle, we obtain a new type of corpuscle with four segments and two mouths, which can be linked to two other corpuscles. It consists of two parts that are only connected at the top and the bottom vertex. Depending on which segment has been removed, there are five corpuscle types, which give rise to different link angles (see Fig. 3). There are also corpuscle types with three or more mouths. For connecting the corpuscle units, we fix a specific orientation: in each unit, we distinguish between a red "top" vertex and a blue "bottom" vertex. Other vertices around a mouth are marked by colours as well (see Figures 2 and 3), and newly attached units must be oriented such that identical colours match.

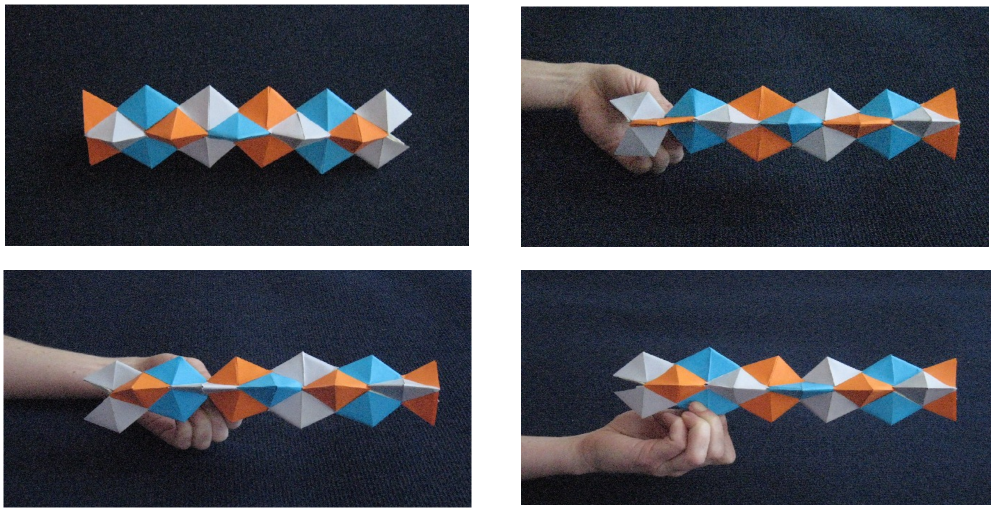


Figure 4: An open corpuscle chain can change its conformation. Top left: corpuscle chain with alternating purple, orange, and blue colours. Top right: as one orange unit is pushed into a flat shape, the other orange units also become rather flat, while nearby blue and orange units become thick. The conformations of individual units are approximately periodic with period 3. Bottom: the same, for purple and blue units.

With this convention, we can describe a chain by the sequence of unit types. In the following, we shall only consider structures consisting of straight units (type (2,2)), weakly curved units (types (1,3) and (3,1)) and weakly curved branch points (type (1,1,1)); the notation is explained in Fig. 3.
Corpuscles with two mouths can be linked to a straight chain (see Figure 3b, top): we start with an open type (2,2) unit and iteratively add more units on the right. We can also construct curved chains: in this case, new units are not added along the previous direction, but at an angle. Simple helices arise, for instance, from repetition of type (3,1) units (see Figure 3b, bottom).

3.2 Collective motion of corpuscle chains

Corpuscle chains can show collective conformation changes: as in the Goldberg icosahedron, the motion in one unit affects the motion of its neighbours. In paper models, the resulting collective motion leads to similar conformations three units to the left and to the right: as one corpuscle is becoming flat, the two preceding and the two subsequent units in the chain simultaneously become thick, while the third, the sixth and so forth, to both sides, become flat. Hence, the conformations along the chain are approximately periodic with period three (see Figure 4), in contrast to the chain itself, which at first sight would suggest a period of 2 (alternating vertical and horizontal units).

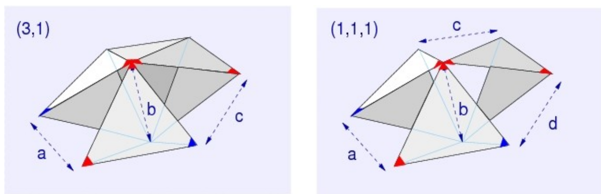


Figure 5: Conformation parameters for individual corpuscle units. Left: in a corpuscle with two mouths, the conformation can be characterised by the width *a* of the first mouth and the thickness *b*. If both values are fixed, the width *c* of the second mouth is determined by a function $c = f(a,b)$. Right: in a three-mouth corpuscle, *a*, *b*, and *c* can be chosen, and d is determined by a function $d=g(a,b,c)$. If this corpuscle is attached to an existing chain, *a* and *b* will be determined by the previous unit, but *c* is a free parameter.

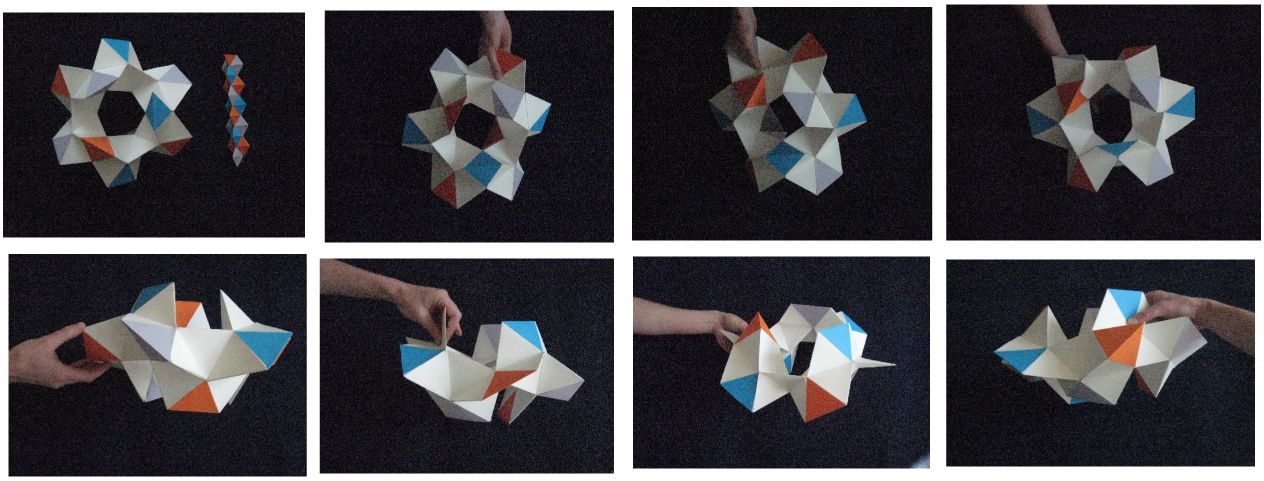


Figure 6: Deformation of the 12-ring. If one of the units is pressed, the entire ring shows a collective conformation change (requiring only little deformation of the triangles). Units are marked by alternating purple, orange, and blue colour as in Figure 4. In the collective motion, units of the same colour (four units per ring) display synchronous behaviour. The pictures show (from left to right) the neutral conformation, orange units flattened, purple units flattened, and blue units flattened.

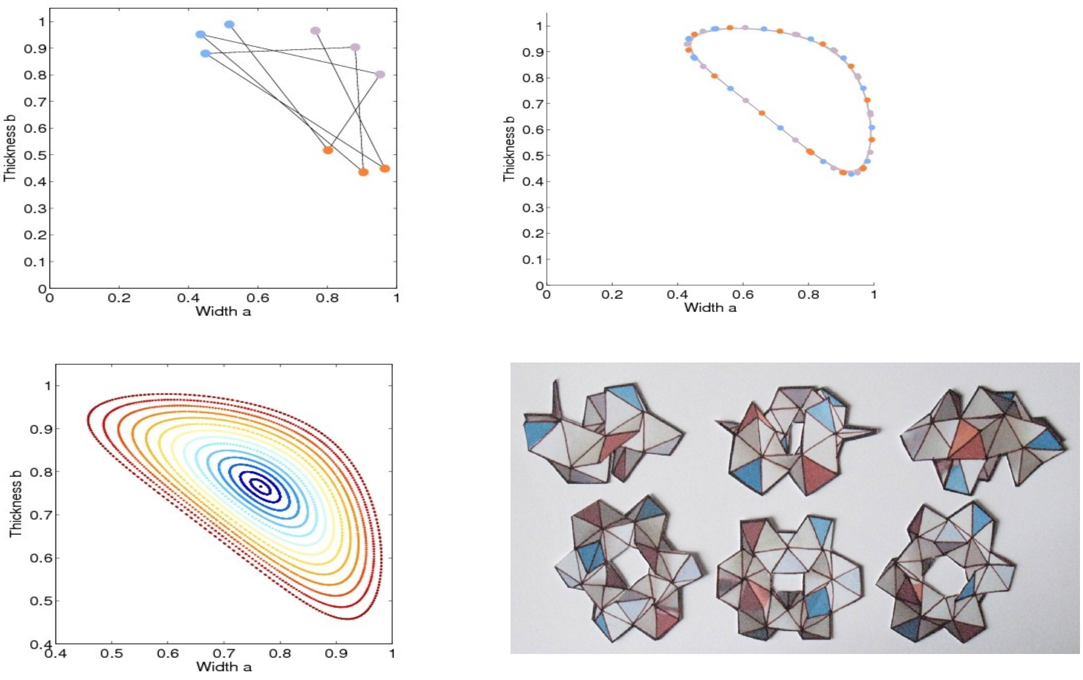


Figure 7: Conformation parameters in the corpuscle chain (see Fig. 4). Each unit is described by its two conformation parameters a_i and b_i (compare Fig. 5). Top left: in the *a*-*b*-diagram, units are shown as dots; lines connect neighbouring units; for the colours, compare Fig. 4. The parameters a_1 and b_1 of the first unit are given(purple dot at about (0.8, 0.95), leftmost unit in the chain): the following conformations resemble each other every three units. Exact periodicity would mean that points of the same colour coincide. Top right: the same plot, with more units shown and similar units connected by lines. All conformations lie on a closed curve in parameter space. Bottom left: the same kind of curve, shown for different choices of the initial parameters a_1 and b_1. Bottom right: conformations with period 3 in the 12-ring (compare Fig. 6).

The approximate periodicity can be described in mathematical terms. The first (leftmost) corpuscle has two mouths; we can fix its conformation by prescribing two parameters, for instance, the width a of the left mouth and the thickness b (see Figure 5). These conformation parameters directly determine the width c of the right mouth. For a corpuscle with two mouths and four segments, the values of c can be computed by

$c = f(a,b) = 2\, r(b) \sin(\pi - \arcsin(a/(2\, r(b))) - 4 \arcsin(1/(2\, r(b))))$

with the corpuscle radius $r(b)=\sqrt{1-(b/2)^2}$.

In the neutral conformation, a, b, and c will show the same value, so we obtain $a \approx 0.7653$ (the solution of $a=f(a,a)$). If we attach a second corpuscle (called "unit 2"), its conformation parameters are determined by the parameters of the first corpuscle ("unit 1"): the left mouth width a_2 of unit 2 equals the thickness b_1 of unit 1, while the thickness b_2 of unit 2 equals the right mouth width c_1 of unit 1. The right mouth width c_2, however, is determined by the function f.

Altogether, the conformation of the first unit - the starting point of the chain - determines the conformations of all other units, so the conformation of the entire chain is determined by two initial conformation parameters a_1 and b_1. Thus, each unit contributes a new parameter, which can be computed from the two previous parameters - just like in the Fibonacci sequence.

In the neutral conformation, all units have the same shape and the chain can be continued indefinitely (point in the centre of Fig. 7, bottom left). For the choice of initial conformation parameters shown in Fig. 7, top, the parameter pairs approximately repeat themselves after three units, forming a closed curve in conformation parameter space.

Deviations from the neutral starting point lead to approximately elliptic curves that are deflected by the boundary of the admissible parameter region $(a \le 1, b \le b_{max} \approx 1.0515)$.

The different curves can be characterised by a single parameter that describes the deviation from the neutral conformation.

The numerical calculations suggest that chains built from a wide range of initial conformations can be indefinitely extended without deformation. This implies, in turn, that infinite chains with exact regular triangle faces are flexible.

3.3 Corpuscle rings

If appropriate sequences of link angles are chosen, the two open ends of a chain come in close vicinity; by slightly deforming the edges, we can join the open ends and obtain a closed ring. In doing so, we distribute the mismatch between the open ends over all units of the chain. This resembles the construction of the tempered scale in music, where the Pythagorean comma (a mismatch between similar tones) is equally distributed over all intervals in the scale.

Figure 8 shows three possible rings consisting of eight, twelve, and sixteen corpuscle units, respectively. The 8-ring in neutral conformation (conformation parameters a=b for all units) closes with high numerical precision. All three rings emerge from weakly curved corpuscles of types (1,3) and (3,1), each consisting of a larger part (three segments, called *bridge*), a smaller part (a single segment), and two mouths in between. Bridges always form the outer side of a curve and are therefore exposed at the exterior side of the rings, while the single segments are located on the inner side.

Can these ring structures also show collective motions? The possible conformation patterns along a chain do not depend on whether the chain is straight or forms a ring. For collective motion in a ring, however, the conformations of the first and the last unit have to be consistent - at least to some approximation. In the 8-ring and 16-ring, the number of units is not a multiple of three, so circular three-phase pulses will not match at the chain ends, so the closed rings are rigid. The 12-ring, in contrast, is easily deformable and shows periodic conformation patterns with period 3, as we see in Figure 6.

(a) (b) (c) (d)

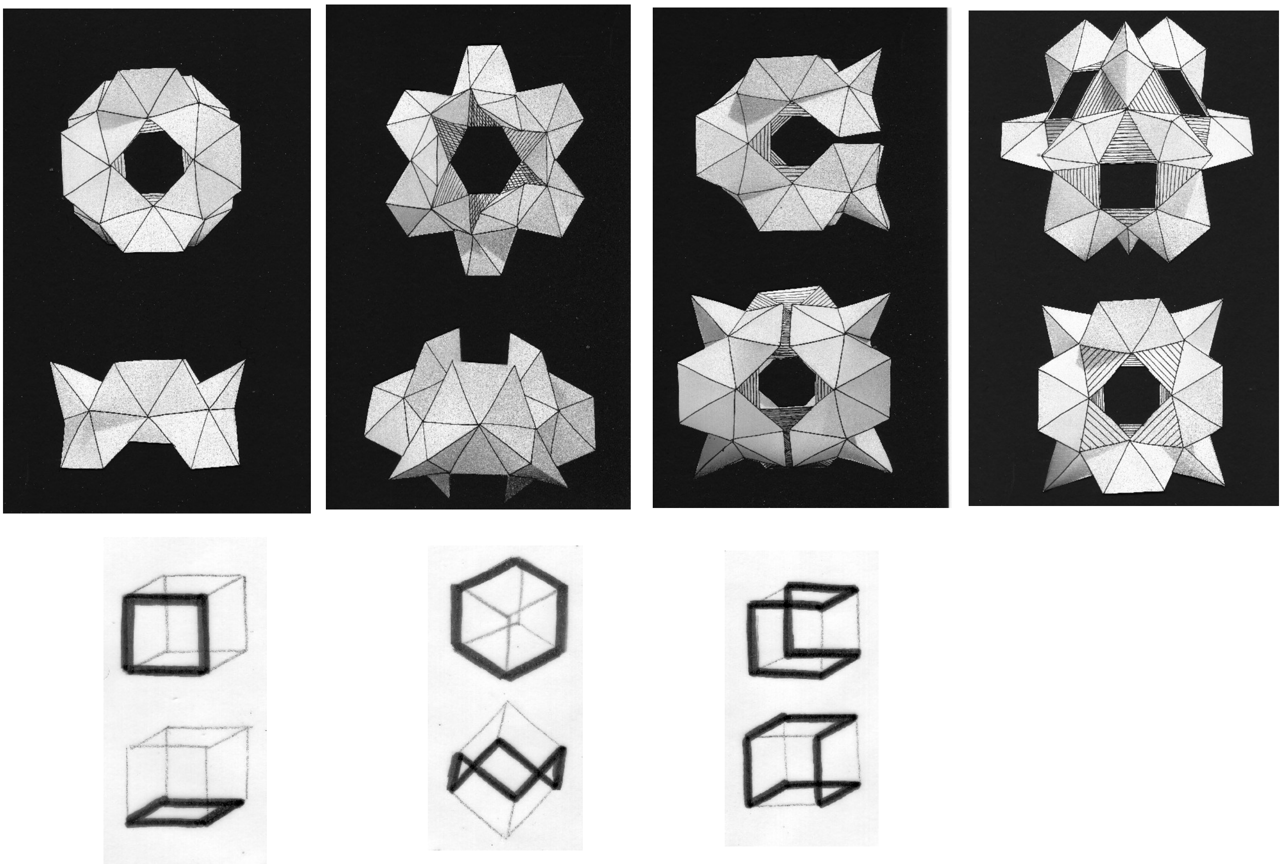


Figure 8: Rings and ball built from corpuscle units. (a) The 8-ring (top and front view) consists of 64 regular triangles, 96 edges and 32 vertices. The inner part contains two hollow squares that are rotated against each other by 45° and connected by a stripe of eight triangles. The outer part is formed by four bridges extending towards the top and four bridges extending towards the bottom. The 8-ring is related to a square (bottom). Each vertex and each edge of the square corresponds to one of the corpuscle units. (b) The 12-ring contains 96 regular triangles, 144 edges and 48 vertices. Its six innermost edges surround a cube. The exterior is formed by three bridges extending towards the top, three bridges towards the bottom, and six bridges towards the outside; they correspond to six vertices and six edges of a cube (bottom). (c) The 16-ring contains 128 regular triangles, 192 edges and 64 vertices. Its eight innermost edges encircle a cube (bottom), corresponding to the eight vertices and eight of its edges. (d) The corpuscle ball. In the scheme on top, the quadratic openings are shaded in black. The corpuscle ball has cubic symmetry and contains 144 regular triangles, 216 edges and 64 vertices.

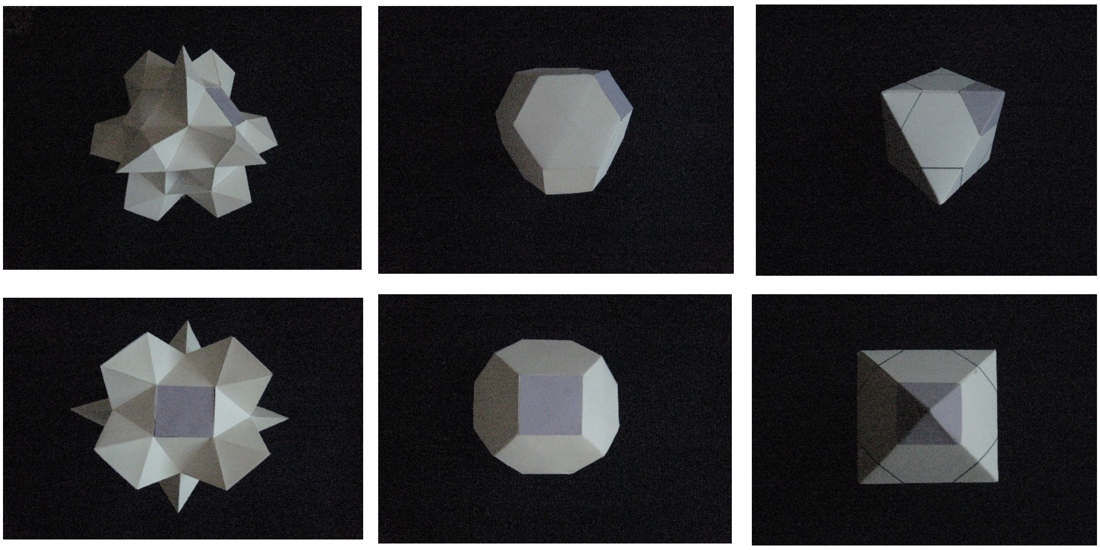


Figure 9: The corpuscle ball (left) can be represented by a truncated octahedron (centre) or by a regular octahedron (right). Left: corpuscle ball seen from its three-fold (top) and four-fold (bottom) symmetry axis. To show the correspondence between corpuscle ball and the octahedra, the quadratic holes are shown here as purple faces. Centre: truncated octahedron. Its six square faces (purple) correspond exactly to the quadratic holes of the corpuscle ball. The hexagon in the truncated ocathedron (TO) corresponds to the flat branch point corpuscle that allows for interlinking two corpuscle balls. Right: by completing a TO with little pyramids, we obtain a regular octahedron, which then also corresponds to a corpuscle ball.

4. CORPUSCLE NETWORKS

Corpuscles of type (1,1,1) can be linked to three neighbours and serve as branch points for building networks. An example is the corpuscle ball, which consists of twenty tightly interconnected units (Fig. 8, left). Corpuscle balls can be interlinked to form three-dimensional periodic patterns.

4.1 The corpuscle ball

In a cube, each vertex connects three edges, while each edge connects two vertices. To obtain a corpuscle ball with cubic symmetry (see Fig. 9), we decorate each vertex of a cube with a branch point unit (type (1,1,1)) and each edge with a bridge unit (type (1,3) or (3,1)); the units fit into each other with little deformation. The corpuscle ball is not flexible since it contains the 8-ring, which by itself is already rigid.

Corpuscle balls can be interlinked in two alternative ways, both along the three-fold axis. Consider the three-fold symmetry centre shown in Figure 9, top left: it consists of a relatively flat branch unit (seen from top) and the three adjacent bridges. Another corpuscle ball can be attached to this vertex such that these four units are shared by both balls. We shall call this configuration of two balls the *even connection*. Alternatively, we can start with an even connection and then rotate one of the balls by 60 degrees: in this arrangement, each ball retains its three bridges, and all six bridges meet edge to edge. The branch point corpuscle is completely covered by the bridges and can be disregarded. This configuration is called the *twisted connection* (see Fig. 10). Hence in both pairwise connections, corpuscles are linked by a common branch point unit (in the even case) or meet edge to edge (in the twisted case). The connection of more than two corpuscle balls in periodic patterns can lead to self-penetration, that is, intersection of triangle faces.

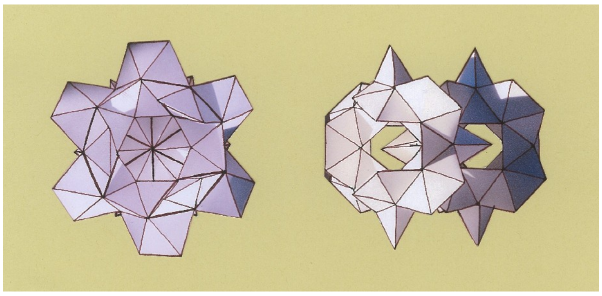


Figure 10: Twisted connection of corpuscle balls. View along the three-fold axis (left) and from the side (right). The twisted connection corresponds to two truncated octahedra joined to each other in the twisted arrangement.

4.2 Representing the corpuscle ball by a truncated or regular octahedron

Complex structures consisting of many corpuscle balls are hard to visualise; for simplifying the following constructions, we represent the corpuscle ball by simple proxy solids, the truncated octahedron (TO) and the regular octahedron (RO) shown in Figure 9. The TO is an Archimedean solid bounded by six squares and eight regular hexagons; it has cubic symmetry and can be obtained by cutting off the corners of a Platonic octahedron. By adding small square pyramids to the square faces, the regular octahedron can be restored. We can arrange our proxies in periodic patterns and then decorate (i.e. replace) them again with corpuscle balls.

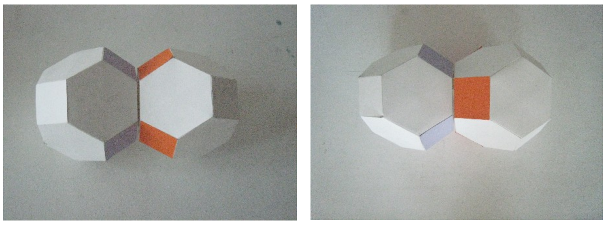


Figure 11: Truncated octahedra can be joined in two different manners, the even arrangement (left) and the twisted arrangement (right).

If two corpuscle balls are connected, the corresponding TO are joined with each other at their hexagonal faces (Fig. 11). The two possible connections between corpuscle balls (even and twisted) correspond to two different ways to join the hexagons. As shown in Figure 11, the hexagon surface joining the two TO is surrounded by squares and hexagons. In the even arrangement, squares from the one component meet squares from the other component; so do the hexagons. In the twisted arrangement, on the other hand, squares and hexagons have to sit next to each other. We can switch between the two arrangements by turning one of the TO by 60°. As shown in Figure 9, right, the corpuscle ball can also be represented by a regular octahedron. If two octahedra, sitting face to face, are decorated with corpuscle balls, we obtain an even connection.

4.3 Periodic corpuscle structures

Our two proxy solids, the TO and the RO, can be arranged in periodic patterns. Figure 12 illustrates a periodic, space-filling arrangement of truncated octahedra. If we decorate them with corpuscle balls, the balls will be linked by twisted connections.

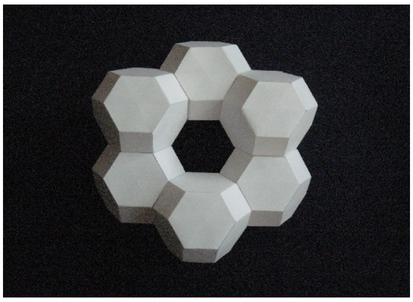


Figure 12: Periodic arrangement of truncated octahedra (TO). Truncated octahedra can be arranged in a three-dimensional periodic, space-filling pattern. If the TO are decorated with corpuscle balls, the twisted arrangement between the TO (compare Fig. 11, right) corresponds to twisted connections between the corpuscle balls (see Fig. 10).

Figure 13, on the other hand, shows a non-space-filling arrangement of octahedra. All of them are joined face to face - corresponding to even connections between corpuscle balls. This structure can be repeated in a cubic lattice.

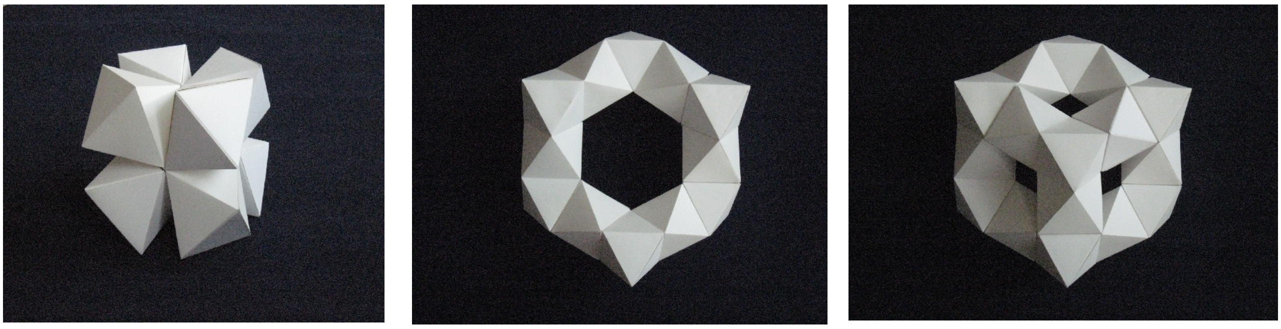
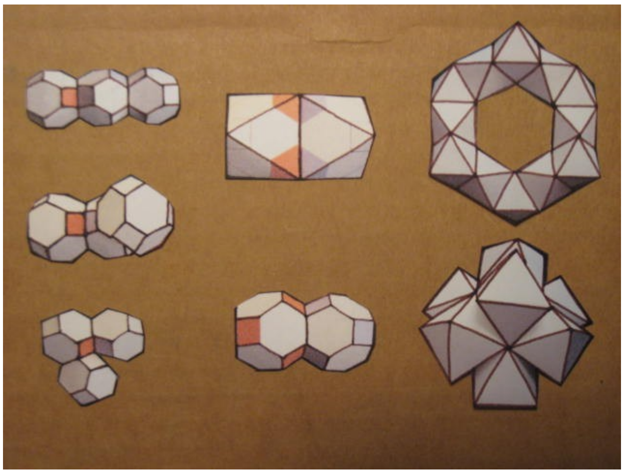


Figure 13: Periodic pattern based on regular octahedra (RO). The structure shown can be decorated with corpuscle balls forming even connections. Left: a regular octahedron is surrounded by eight octahedra attached to its faces. When this structure is inscribed into a cube, it can be repeated periodically in a cubic lattice. The cube vertices will be surrounded by octahedral holes: by filling these holes again with octahedra, we obtain a periodic RO structure with cubic symmetry. It consists of intertwined octahedra chains ("filaments") that point along four directions in space. The remaining empty spaces form a periodic corpuscle network made of four-fold symmetric corpuscles of type (1,1,1,1) (not shown). Centre: the periodic RO structure contains rings of 12 octahedra. Each ring contains six overlapping filament pieces (length 3, two pieces for each of three filament directions, ring axis parallel to the fourth filament direction). Right: by adding more octahedra to the ring, we obtain a structure resembling a parallelepiped. Each edge is formed by a filament piece of length 3.

(a)



(b)

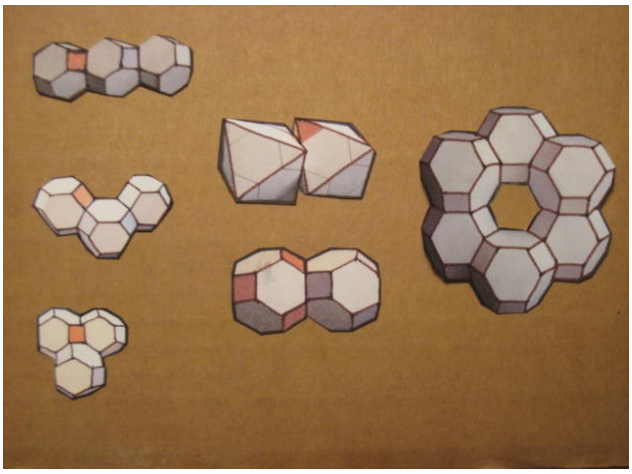


Figure 14: The two arrangements of octahedra: (a) Even arrangement. Left column: even arrangement of three TO in a straight line, with a weak angle, and with a strong angle. Centre column: even arragement of truncated (top) and regular (bottom) octahedra. Right column: building blocks for octahedra networks (see Fig. 13). (b) Twisted arrangement of truncated octahedra. Left column: twisted arrangement of three TO in a straight line, with a weak angle, and with a strong angle. Centre column: twisted arragement of truncated (top) and regular (bottom) octahedra. Right column: periodic space-filling arrangement of truncated octahedra (Fig. 12).

The two arrangements - regular octahedra with even connections and truncated octahedra with twisted connections, are shown again in Figure 14. In both cases, the resulting periodic structures can contain self-intersections between some of the corpuscle units.

5. CONCLUSIONS

The corpuscle is a simple basic shape consisting of regular or almost regular triangles. It allows to build a variety of three-dimensional structures, including the Goldberg icosahedron, chains, rings, and the corpuscle ball with cubic symmetry. Some of the structures are flexible or easily deformable. With a convention about the relative orientation corpuscle units, complicated chain structures can be described by a numerical short notation. Periodic three-dimensional networks can be formed by decorating regular or truncated octahedra, with corpuscle balls.

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