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TENSION AND DEFORMATIONS IN ELASTIC POLYHEDRAL RINGS MADE OF CORPUSCLE ELEMENTS

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ABSTRACT: Corpuscles are compact geometric structures formed by regular triangles with edges functioning like hinges. They can be connected to form chains, closed rings, and spatial networks. Some of the compound structures are flexible. Their geometric shape can be varied continuously and movements of a single element lead to collective conformation changes along the corpuscle chain. In many cases, however, this requires a slight deformation of the edge lengths. Open-ended corpuscle chains can be fully flexible and the element conformations repeat each other approximately after three units. We also observed that ring-like chains made from paper can be deformed more easily if the number of ring elements is a multiple of three. To investigate this further, we consider a number of corpuscle structures and simulate their deformations by a numerical model. Edges are modeled as elastic springs and the structures are relaxed towards conformations of minimal energy. We find that most of the corpuscle rings can only be closed if some tension is applied. Moreover, the computer experiments suggest that if this tension is high, some of the elements will spontaneously be punched in and the structure will lose its symmetry. In both of the striking examples we found, the ring size was in fact a multiple of three.

Keywords: Corpuscle, flexible polyhedron, symmetry breaking, elastic deformation

1. INTRODUCTION

Corpuscles are flexible geometric structures formed by regular triangles. Like in a paper model, the edges function as hinges. Some of the adjacent triangles are not joined by an edge, but by a slit, which creates open ends (“mouths”) at which two corpuscles can be joined. A simple example is Goldberg's “Siamese dipyramid” icosahedron [1], which consists of two elements, each containing five triangle segments and one mouth. When joined, the two elements still show coupled movements: if one element has a thick shape (long central axis, narrow mouth), the other one is flat (short axis, wide mouth).

Several corpuscles with two mouths can form straight or curved chains, while elements with three mouths can serve as branch points in extended networks. Some of the chains are strongly curved and can be closed to form rings [2]. Paper models, which are slightly deformable, indicate that the chain deformations are approximately periodic with a period of three and that rings made from multiples of three elements can be deformed, while other rings tend to be stabilized. The 8-ring, for instance, is rigid, while the elastic 12-ring can be deformed easily, showing a repeated sequence of flat/bold/bold elements. Moreover, a paper model of a new 6-ring flips spontaneously into a non-symmetric conformation, which apparently reduces the tension in the material. However, it is hard to tell from paper models alone how much edge deformation is needed to obtain a closed ring or to deform it. Previous calculations have confirmed that open-ended chains can show flexible collective deformations, which are approximately periodic with a period of about three [2]. But it is still unclear which of the corpuscle structures can be build exactly (that is, without deforming the edges) and which would be the preferred conformations for the others. To test this, we now studied corpuscle structures of different size, including rings consisting of 6, 8, 10, 12, and 60 segments, and simulated their stable conformation and possible deformations by a numerical model. In the model, edges are modelled by elastic springs following Hooke's law. We studied how much the edge have to be deformed to build these structures, whether their stable conformations show any symmetry breaking, and what are their softest deformation modes.

2. CORPUSCLE STRUCTURES

Corpuscles are composed of *segments* (two triangle faces connected by a flexible edge) and *mouths* (segments in which the faces have been omitted and where the mouth of another corpuscle element fits in), all arranged around a central axis [2]. The elements can be denoted by numbers of connected segments: in a (2,2) corpuscle element, for instance, there are two pairs of segments, separated by two mouths. A (1,3) element is composed of a single segment, a mouth, three segments, and another mouth. Other types of elements are named accordingly. Corpuscles can be connected to build a variety of structures. Apart from the ones presented in [2], we consider here a 44-hedral cluster, a cluster of four elements, and new rings containing 6, 12, or 60 elements. All structures are shown in Figure 1 (as paper models) and in Figures 3a, 3b, 3c, and 3d (as computer models). Their topologies are listed in Table 1.

3. ELASTIC SPRING MODEL: STABLE CONFORMATIONS AND SOFT DEFORMATION MODES

Together with the Goldberg icosahedron, our list comprises corpuscle clusters and rings of 2, 4, 6, 8, 12, 16, and 60 elements, as well as the cube-like corpuscle ball (20 elements) and a 44-hedral cluster. Most of these structures require elastic edges, which can change their length. We have studied their conformations by a numerical model in which edges are elastic springs following Hooke's law. Each conformation is scored by an energy resulting from stretching and compression of the edges. For each structure, we have computed a conformation of minimal energy ("stable conformation") and determined its symmetry. Next, we determined possible low-energy deformations around this stable conformation from the eigenvectors of the energy function's Hessian matrix.

For the calculations, a corpuscle structure is represented by a set of nodes (with indices α and coordinate vectors $(x_{1\alpha}, x_{2\alpha}, x_{3\alpha})^T$) and by a set of edges given as ordered node pairs. The total energy reads $E = 1/2 \, \Sigma_{(\alpha,\beta)} \, (D_{\alpha\beta} - L_{\alpha\beta})^2$, where $D_{\alpha\beta}$ is the Euclidean distance between nodes α and β and $L_{\alpha\beta}$ is the nominal edge length between them. Usually, we consider a standard edge length $L_{\alpha\beta} = 1$. The sum runs over all pairs (α, β) of nodes joined by an edge and satisfying $\alpha < \beta$.

To obtain a closed ring, we start from an open chain, determine nodes at the open ends to be matched, and deform the structure such as to bring these nodes in close vicinity. Then, we collapse the matched nodes and numerically relax the structure further to obtain a stable conformation, that is, a local energy minimum. If all edge lengths in this conformation are equal to 1 within numerical accuracy, the structure can be built exactly without edge deformation.

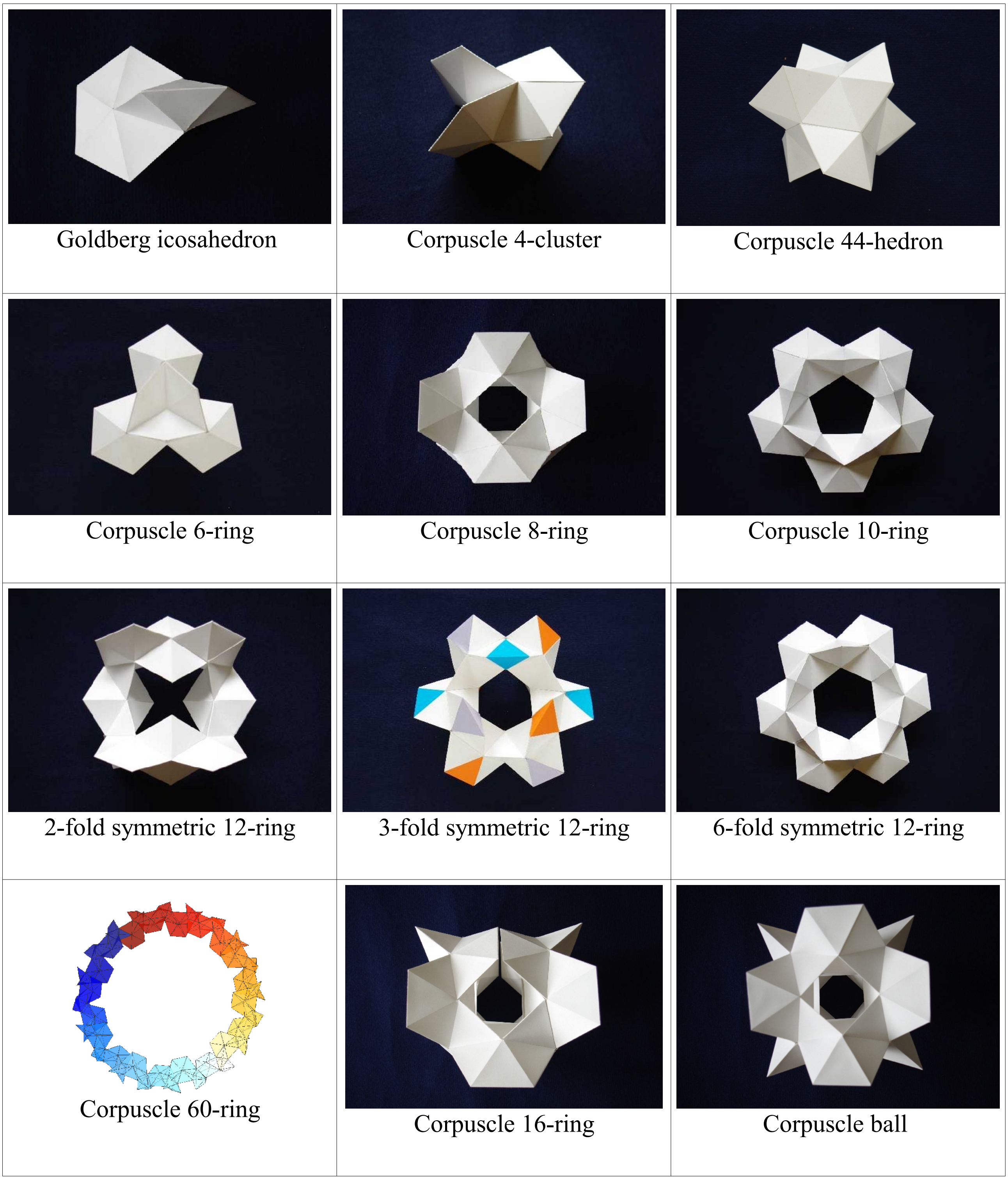

Figure 1: Corpuscle structures shown as paper models. Details on the topologies are listed in Table 1.

Name	Elements	Nodes	Edges	Triangles	Rotation symmetry	Mean edge energy	Symmetry broken	Exact
Goldberg icosahedron	2	12	30	20	2	appr. 0		x
Corpuscle 4-cluster	4	18	48	32	2	appr. 0		x
44-hedral cluster	6	24	66	44	2,3	appr. 0		x
Corpuscle 6-ring	6	25	72	48	3	appr. 0	x	
Corpuscle 8-ring	8	32	96	64	4	appr. 0		x
Corpuscle 10-ring	10	40	120	80	5	$8.5 * 10^{-5}$		x
Corpuscle 12-ring (2-fold)	12	48	144	96	2	$6.4 * 10^{-5}$		
Corpuscle 12-ring (3-fold)	12	48	144	96	3	$3.9 * 10^{-5}$		
Corpuscle 12-ring (6-fold)	12	48	144	96	6	$7.6 * 10^{-8}$	x	
Corpuscle 60-ring	60	240	720	480	10	$3.1 * 10^{-8}$		
Corpuscle 16-ring	16	56	188	128	2	$3.3 * 10^{-5}$		
Corpuscle ball	20	64	216	144	4	$2.4 * 10^{-5}$		

Table 1: Corpuscle structures studied. For the geometric shapes, compare Figures 1 and 3a-d. The first columns indicate the numbers of elements, nodes, edges, and faces. The first three structures ("clusters") satisfy Euler's formula (#nodes + #faces = #edges + 2) for convex polyhedra, while all others, due to their differing topologies, have Euler's characteristics different from 2. "Rotation symmetry" refers to an idealized geometric shape of maximal symmetry, which may show unequal edge lengths and need not represent a stable conformation in the elastic spring model. The last three columns summarize results from the numerical calculations. Average edge energies (mean square deviation from the nominal edge length of 1) were computed after relaxing the structure to a stable conformation. Values below 10^{-9} are labeled as "appr. 0"; larger values suggest that the structure cannot be formed with equal edges. Structures are labeled as "broken symmetry" if the stable conformation breaks the symmetry of the graph (as determined by visual inspection and by multiplicities of edge lengths). A structure is labeled "exact" if there exists a symmetric stable conformation with equal edge lengths (again, within numerical accuracy).

Next, we studied the soft deformation modes. Based on the stable conformation obtained, we computed the Hessian of the energy and determined its eigenvalues and eigenvectors. For these calculations, we considered two alternative starting points: (i) we assumed that all edges have a nominal length; (ii) we assumed that each edge's nominal length is the one found in the stable conformation. The results were similar, but the second alternative excludes, by construction, the possibility of negative eigenvalues due to numerical inaccuracy. In the following, we shall refer to the second possibility. In any case, translation and rotation of the entire structure are energy-neutral and lead to six zero eigenvalues, which we could recover with good numerical accuracy. The next smallest eigenvalues are associated with soft elastic deformations.

4. DEFORMATION AND SYMMETRY OBSERVED IN CLUSTERS AND RINGS

Paper models and simulation results are shown in Figures 2a-e and 3a-d, respectively. The numerical results are listed in the last three columns of Table 1 and soft deformation modes are shown at www.korpuskel.de as movies. In this section, we shall describe the structures' behavior in more detail.

4.1 Corpuscle Clusters

The first three structures (called "clusters") can be built with rigid edges. As shown by Goldberg [1], the Goldberg icosahedron has three conformations in which triangles stay regular. A paper model moves continuously and with little effort between these conformations. This "breathing" movement also appears as the softest deformation mode in the calculations.

The four-corpuscle-cluster consists of four units, each with four segments and one mouth. They are assembled in pairs around a non-regular tetrahedron, just like the Goldberg icosahedron. According to the simulations, its softest deformation mode also resembles the breathing of the Goldberg icosahedron, with the two halves opening and closing in opposite phase.

The 44-hedron contains six corpuscles with three segments and one mouth each. These *bridges* lean over the surface of a core solid, which resembles a regular icosahedron. In fact, it represents one phase of B. Fuller's Jitterbug, a continuous movement between a regular octahedron, an icosahedron, and a cuboctahedron. The softest deformation mode of the 44-hedron consists of an extension along one of the main axes. Due to its symmetry, this mode can appear in three different directions.

4.2 Corpuscle rings

Our present selection focuses on rings that can be closed with little distortion, show symmetry, and are possibly deformable.

The 6-ring emerges from alternating (2.2) and (0,4) units. The (2,2) units share the center point of the ring as a vertex. In the paper model, the structure is flexible: its symmetric shape seems to be unstable and flips into a conformation in which one of the (2,2) unit diminishes its volume. So does the (0,4) unit on the opposite side, while the other four elements simultaneously increase their volume. The vertical (2,2) unit keeps some of its volume when the co-acting horizontal (0,4) unit is already flat. This can happen in three different orientations. This behavior is also reflected in the computer model: in the stable conformation, one of the vertical elements is punched in and completely flat, which decreases the overall tension to a very small value and obviously breaks the 3-fold rotation symmetry.

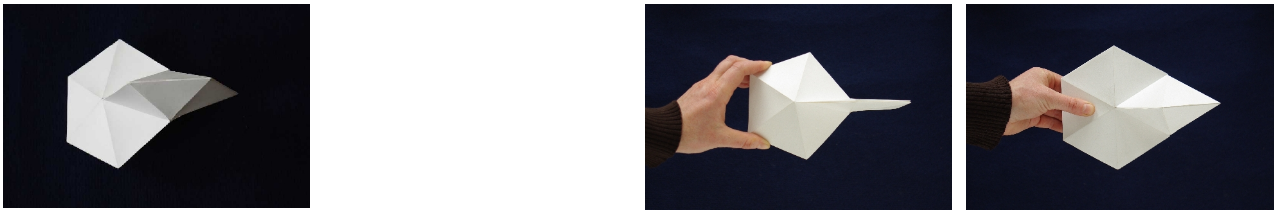

Figure 2a: Deformation of the Goldberg icosahedron paper model. Shown are the symmetric conformation (left) and the two extreme deformations (right).

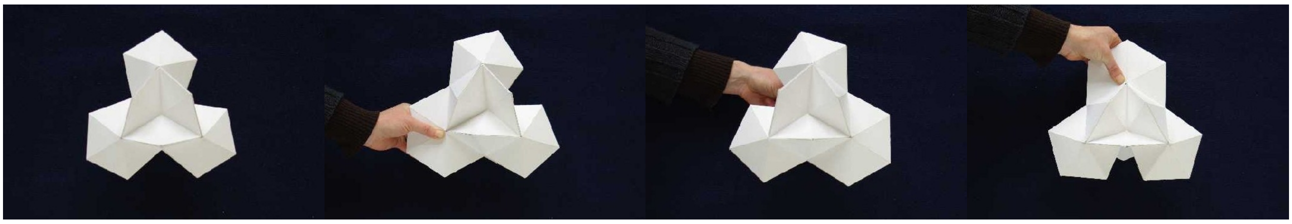

Figure 2b: Deformations of the 6-ring paper model. Shown are the symmetric conformation and three extreme deformations, which have the same shape and differ only in their orientation.

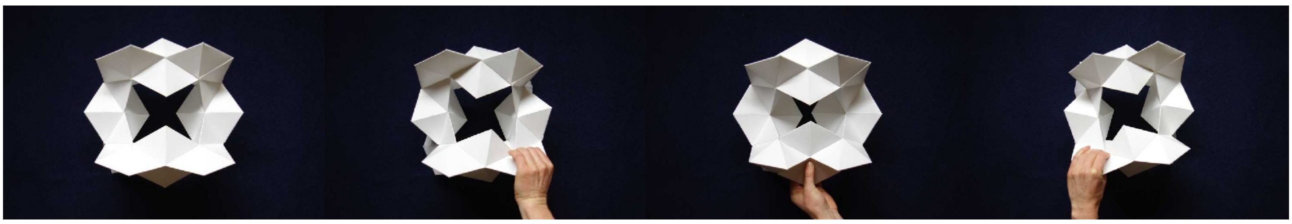

Figure 2c: Deformations of the 2-fold symmetric 12-ring paper model. Shown are the symmetric conformation and the three extreme deformations. There is one outstanding deformed conformation (mid-right) in which all (2,2) type units are flat. The other two extreme conformations (mid-left; right) are mirror images of each other.

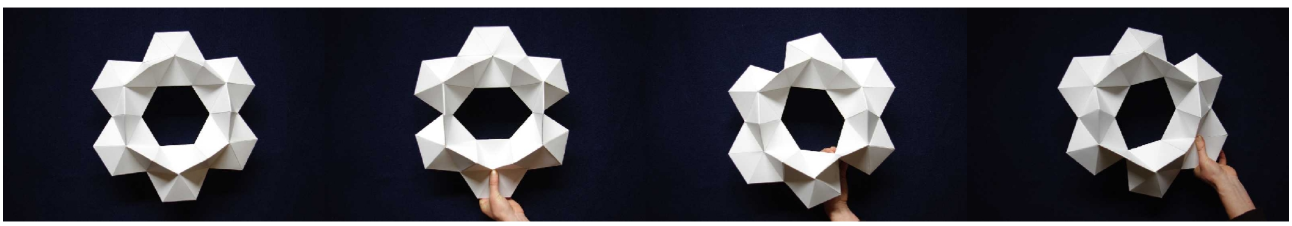

Figure 2d: Deformations of the 6-fold symmetric 12-ring paper model. Shown are the three extremes of deformation and the symmetric conformation. The three deformations can be transferred into each other by rotating the model.

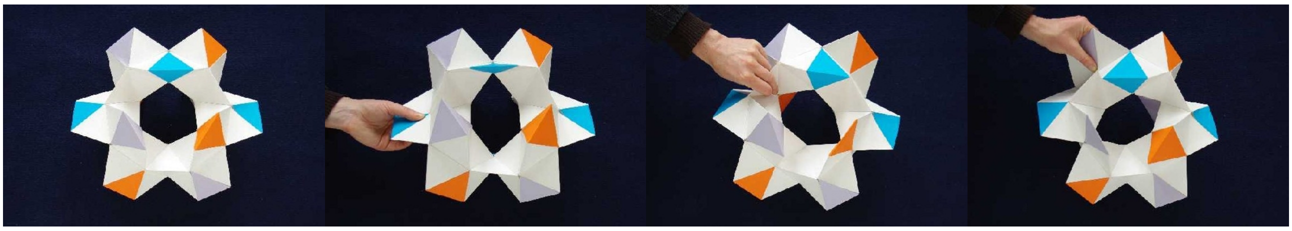

Figure 2e: Deformations of the 3-fold symmetric 12-ring paper model. Shown are the three extreme deformations and the symmetric conformation (left). Units co-acting in the same set are marked by the same color. By switching colors, the three deformations can be transferred into each other by rotating the model.

The 8-ring consists of alternating (1,3) and (3,1) type elements and is the narrowest ring that can be built from such units. The empty area in its center forms a square antiprism. The 8-ring can be closed without deformation and shows hardly any flexibility in the paper models. The 10-ring consists of alternating elements of type (2,2) and (1,3). Five of its elements form a central pentagon. Its stable form is under tension and hardly flexible.

Our three 12-rings have the same number of nodes, edges, and triangles, but differ in their chain sequences and show 2-fold, 3-fold, and 6-fold symmetry, respectively. In the 2-fold symmetric 12-ring, four units of type (2,2) are assembled with four units of type (1,3) and four units of type (3,1). Deformation of this ring creates extreme shapes (see Figure 2c), since the co-acting sets of corpuscles do not gather units of the same type. One set consists of four units of type (2,2), and a flattening of this set causes a contraction of the ring's entire shape. The second set consist of units of type (1,3), and the third set of co-acting subunits are type (3,1). A flattening of these sets causes a twist in the ring's shape.

The 3-fold symmetric 12-ring, in contrast, assembles (1,3) and (3,1) units in alternating order. The inmost edges of six elements form a band meandering up and down three times. This band can be seen as part of a cube in the ring's center space. The structure closes with little tension and can be easily deformed, moving from the equilibrium into three extreme conformations. In each of them, two (1,3) and two (3,1) units become flat.

The 6-fold symmetric 12-ring is built from alternating elements of type (2,2) and (1,3) and surrounds a hexagon. In the paper model, the ring is harder to deform than the 3-fold 12-ring. On the way from its symmetric shape to one of the three extreme positions, four of the elements become flat: two (2,2) units and two (1,3) units. After a first strong effort, the structure seems to reach a more relaxed shape. In the calculation, this ring undergoes a spontaneous symmetry breaking that leads to to the same shape: four segments, in a distance of three segments each, become very thin and partially punched in.

The 60-ring is formed by a sequence of (1,3) type and (3,1) type units alternating after each third element. During the relaxation, its edge tension achieves a very small value. We therefore expect that the 60-ring can be built with edge lengths very close to, but not equal to one. From the same sequence, we can also build rings of other size, but it takes much more tension to close them. The 18-ring, for instance (not presented here), shows a similar spontaneous deformation as the 6-fold symmetric 12-ring.

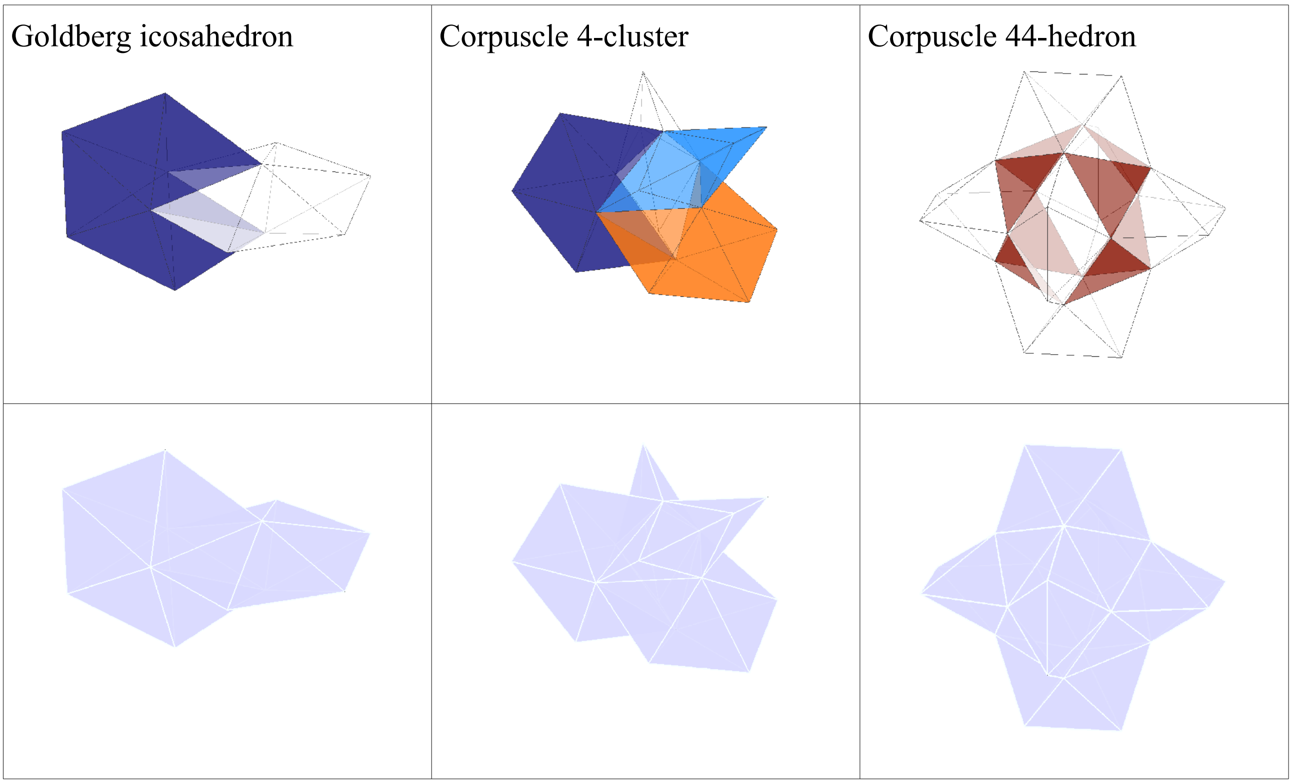

Figure 3a: Corpuscle clusters studied. Details on their topologies are given in Table 1. All edges have their natural length.

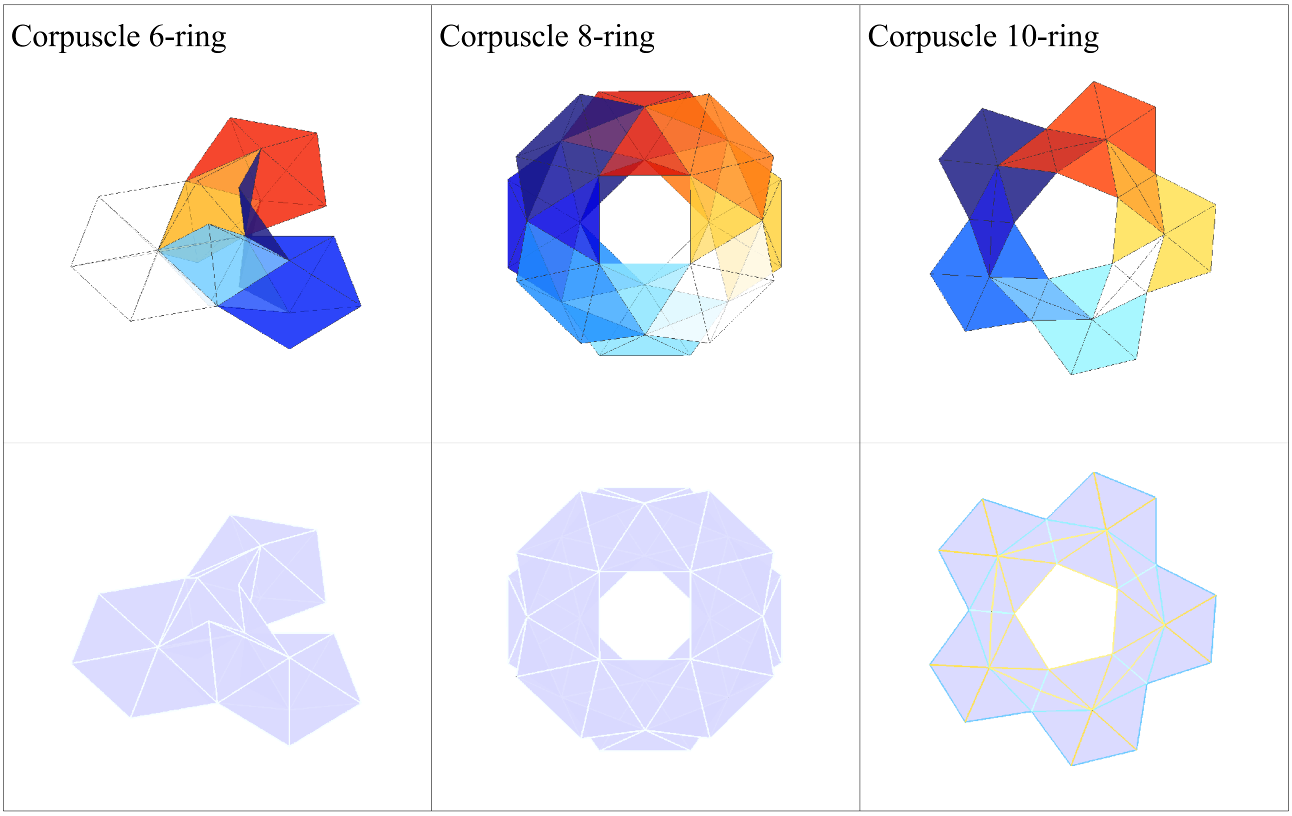

Figure 3b: Three of the corpuscle rings studied. Details on their topologies are given in Table 1. On bottom, edge tension is marked by colors (red: compression; blue: extension).

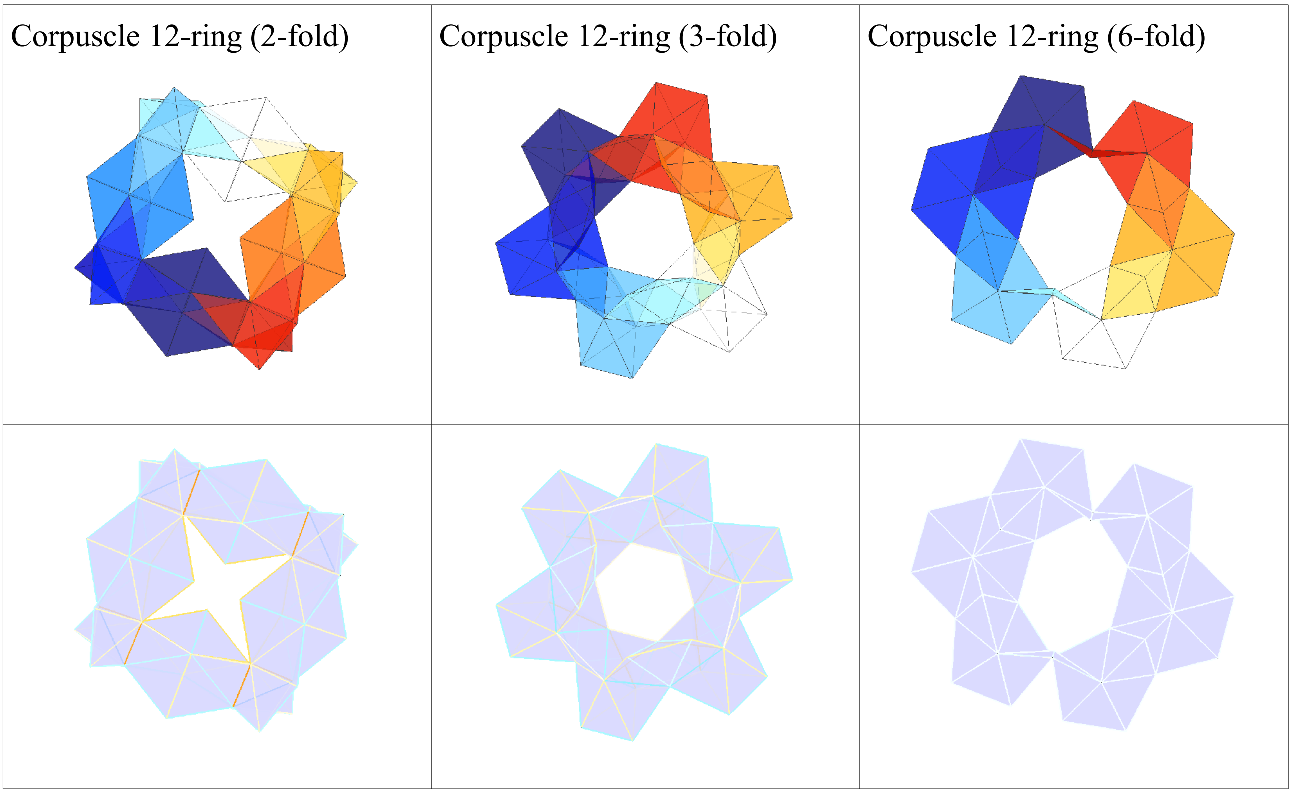

Figure 3c: The three 12-rings studied. Details on their topologies are given in Table 1. On bottom, edge tension is marked by colors (red: compression; blue: extension).

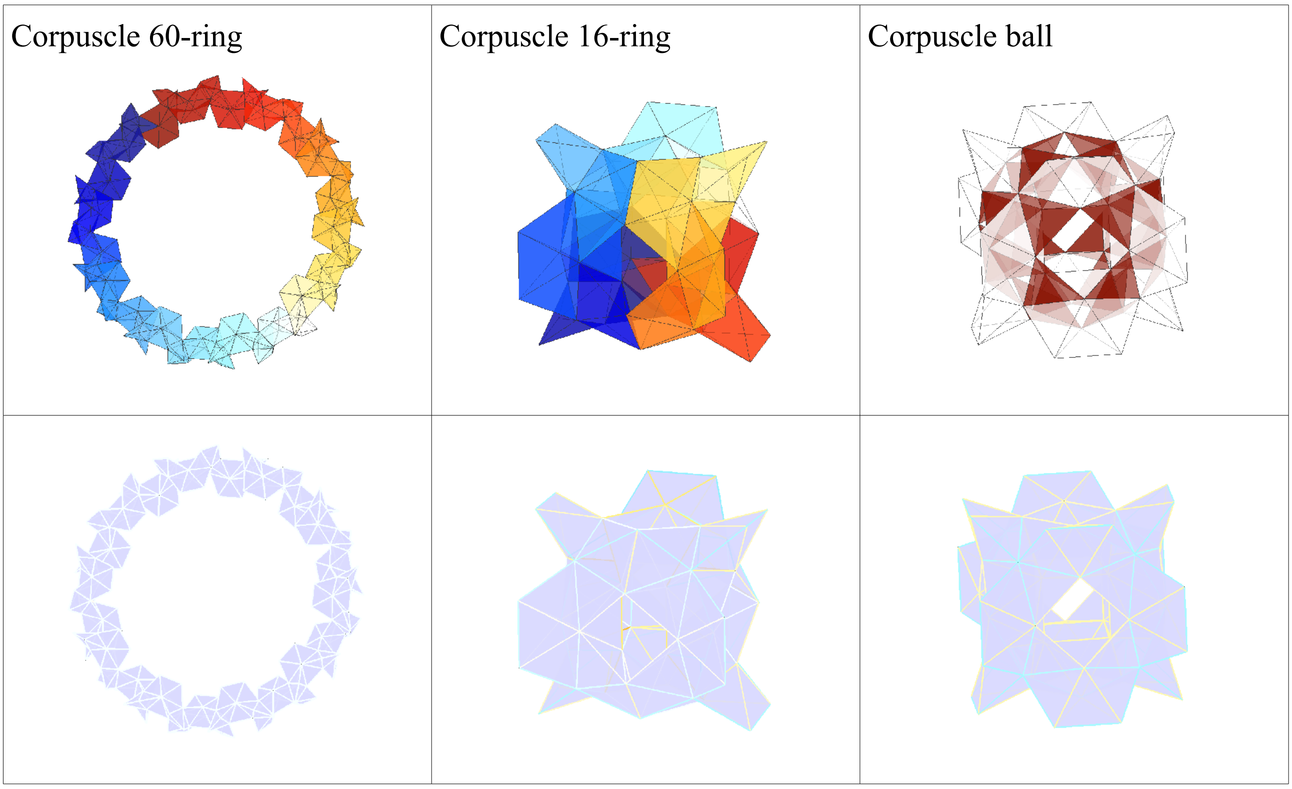

Figure 13: Two rings and the corpuscle ball. Details on their topologies are given in Table 1. On bottom, edge tension is marked by colors (red: compression; blue: extension).

The 16-ring emerges from elements of type (1,3) and (3,1), being repeated after the first – first – second... element. This ring surrounds a cube in its central area, fitting along eight of the cube's edges and covering all its vertices. By adding four bridges, it can be completed to the corpuscle ball, which has cubic symmetry [2]. Both structures show hardly any flexibility.

5. CONCLUSIONS

Our computer experiments suggest that the three clusters (Goldberg icosahedron, 4-cluster, and 44-hedron) can be built with rigid edges, while the rings can only be closed by applying some tension - the 8-ring being the only exception. If the overall tension is high, it need not be distributed evenly over the entire structure, but some of the corpuscle elements may become flattened or even punched in, which spontaneously breaks the overall symmetry. We have seen examples of this in the 6-ring, in the 12-ring with 6-fold symmetry, and also in rings with much stronger tension (18-ring and 24-ring), which were not presented here. Although the selection of structures studied here is far from being comprehensive, the results so far agree with our expectation that deformations can arise more easily if the ring size is a multiple of three.

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ABOUT THE AUTHORS

1. Eva Wohlleben studied sculpture at the Kunsthochschule Berlin Weißensee. In her artistic work, she explores geometrical structures and gives them sensual reality. At her exhibitions, the visitor is invited to touch, move, change and understand geometric structures. Her main interest are monades with a definite flexibility, for which she coined the term "corpuscle geometry". Her current research is focused on duality of irregular polyhedra.

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