Duality in Non-polyhedral Bodies Part I: Polyliner

Eva Wohlleben(✉)

79737 Herrischried, Germany

www.korpuskel.de

Abstract. Duality occurs in pairs of polyhedra, for example between the icosahedron and the dodecahedron, between the cube and the octahedron, and self-dually in the tetrahedron. In this paper, the principle of duality is generalized and linked to the construction of alternating knots. In a polyhedron, each surface has at least three vertices and there are at least four surfaces. There is always a dual polyhedron. From both, an alternating knot (or link) can be constructed, one that has as many crossing points as the polyhedron has edges. It turns out that this result also applies to bodies whose surfaces have less than three vertices and which consist of less than four faces. The resulting bodies can not be assembled like polyhedra from flat faces. In the following, they will be referred to as "polyliner". If each facet has at least two vertices, an alternating knot can be constructed as before. If facets with only one point are present in the polyliner, the construction of the associated knot at this point results in a loop that can be unknotted. If the disentangling is not done, but the crossing point is maintained, then the resulting spatial curves can be cataloged according to their topology.

Keywords: Duality · Polyhedron · Polyliner · Dual graph · Medial graph Kernel · Hull · Bundle · Knot · Link · Unknot

1 Duality

Introduction The described relationships are a classic result of knot theory, where they are represented in a two-dimensional manner, using the terms graph, medial graph, and reciprocal graph. Equivalent representations are given at the end of this article (paragraph 3.3, Fig. 15). The paper is not based on these two-dimensional representations but proceeding from another embodiment of duality: the platonic solids. Focusing on the three-dimensional representatives, the principle of duality is found to rule a wide range of bodies. As solid interpretations of spherical graphs, here they are introduced as "polyliner".

Furthermore, in the spacial inspection the medial graph appears as a discontinuous element during a continuous process of transformation (of a graph in its dual) and is complemented by a second form of mediation, which is here introduced as "bundle".

L. Cocchiarella (Ed.), *ICGG 2018—Proceedings of the 18th International Conference on Geometry and Graphics*, Advances in Intelligent Systems and Computing 809, pp. 484–499, 2019. https://doi.org/10.1007/978-3-319-95588-9_40

1.1 Duality in Form

The Platonic bodies occur in pairs: Each two belong together and complement each other, as seen in the number of components in the solids: The icosahedron has 12 vertices and 20 faces, and its dual partner, the dodecahedron, has 12 faces and 20 vertices. However, the spectrum of bodies that can be dualized goes far beyond the regular polyhedra. In the first place, in addition to the Platonic bodies, all irregular polyhedra are dualizable.

Duality extends into the details of the polyhedral construction, the roles of face and vertex are reversed in a specific way: the trigonal faces of the octahedron are represented in the cube as three-fold vertices, and the tetragonal faces of the cube in the four-fold vertices of the octahedron, where four faces meet. The number of edges remains the same, but a line that connects two points becomes a line that separates two faces and vice versa (Table 1).

Table 1. Topological relationship of faces, vertices and edges in dual polyhedra

Polyhedron A	Polyhedron B
n faces m vertices face types p, q, r … vertex types e, f, g … edge x connects vertices edge y separates faces	n vertices m faces vertex types p, q, r … face types e, f, g … edge x separates faces edges y connects vertices

The fact that every polyhedron has exactly one dual partner is not a metric property but a topological one. In the following, what will be decisive is the number and relative position of faces, edges and vertices.

Examples: Pentahedra It is possible to differentiate between two pentahedra (five-sided polyhedra). One, the "trigonal prism" (Fig. 1), gathers three tetragons around two trigons, and the other one, the "tetragonal pyramid", surrounds one tetragon by 4 trigons.

The 6 vertices of the trigonal prism—all threefold—are translated in the dual partner into 6 triangular faces. Together they form the "trigonal dipyramid". The 5 vertices of the tetragonal pyramid—of which 1 is fourfold, 4 threefold—are translated into 5 faces in the dual partner—1 tetragonal, 4 trigonal. Together they form a tetragonal pyramid, which is upside down from the original.

All pyramids are "self-dual" in this upside down way; other polyhedra show self-duality in different orientation (Fig. 2 column 4, 5).

Hexahedra—Chirality in Polyhedra Respectively, hexahedra with five to eight vertices correspond to penta-, hexa- hepta- and octahedra (Fig. 2).

In the group of hexahedra the phenomenon of chirality in polyhedra occurs, meaning, that two bodies can be identified by a mirror operation. To name the order of the group of hexahedra—the number of its members—a decision has to be made.

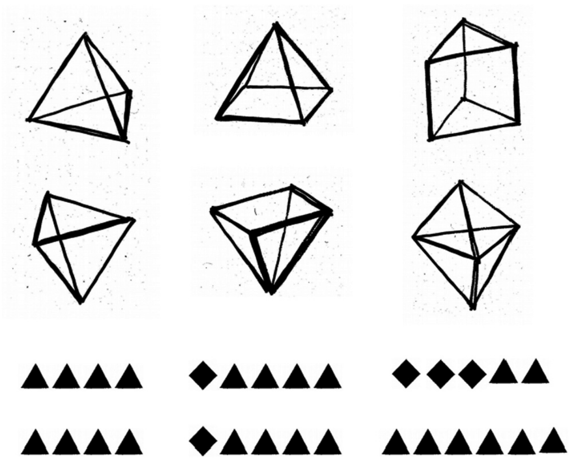

Fig. 1. Tetrahedron and pentahedra with dual partner. From bottom to top: face types of B; face types of A; polyhedron B; corresponding polyhedron A

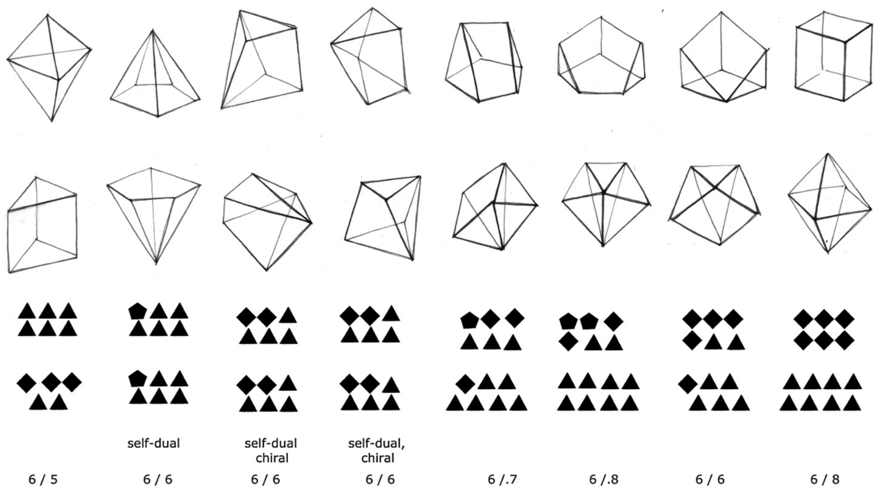

Fig. 2. Hexahedra. From bottom to top: number of faces A/B; properties, types of faces B, types of faces A, polyhedron B with 6 vertices, polyhedron A with 6 faces

Based on the idea that mirror-like polyhedra are as good as identical, one can count the two chiral hexahedra in Fig. 2 as one and the same, so that the overall count comes to seven hexahedra.

1.2 Synthesis

Truncation Dual partners can be translated into each other, by truncating their vertices. On the one hand, this process can be seen as continuous, in that the cutting surfaces, which blunt the corners more and more, move continuously towards the center of the body. On the other hand, this process contains a discontinuous moment. Exactly at its mid-point a special shape is created, which in terms of the number of its vertices and type of its faces is simpler than all the others—the kernel of the dual pair (Figs. 3 and 4).

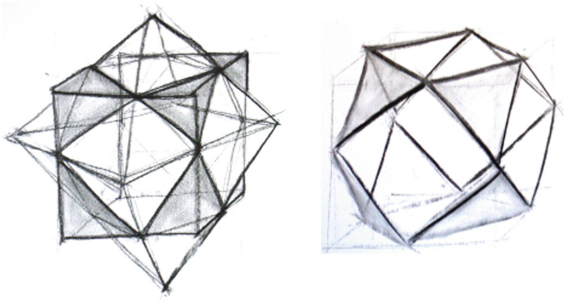

Fig. 3. Intersection—mid-point of transformation process. Left: cube and octahedron in a relative size so that all their edges meet in pairs; right: kernel—their shared volume: the cubeoctahedron

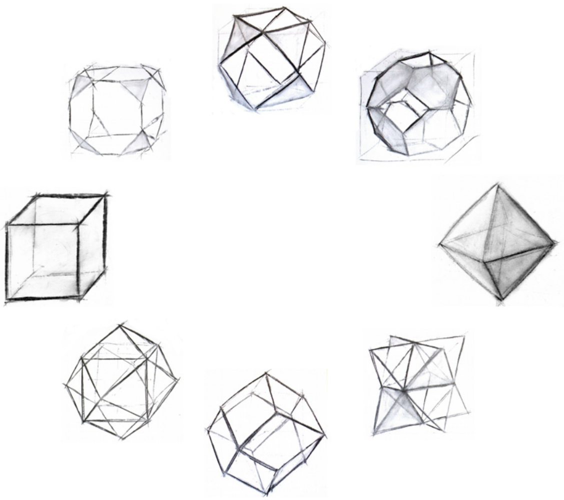

Fig. 4. Transformation circle of truncation and epicuration between cube and octahedron. Clockwise: cube, truncated cube, cubeoctahedron (kernel); truncated octahedron, octahedron, epicurated octahedron, rhombic dodecahedron (hull) and epicurated cube

Intersection From a more harmonious viewpoint truncated forms result from an intersection. Starting with both partners gathered around a common centre point and changing gradually their relative size, in the volume they share transition forms are created: when edges of A and B meet in pairs in one point then the kernel occurs; if one body is smaller then truncation forms are created, and when its vertices lie in the partners surface the intersections common shape equals the smaller initial body.

1.3 Duality in Transformation Process

Not only forms, processes are also dualizable. Truncation corresponds to "epicuration". Table 2 compares the topological changes happening (1) at the beginning of the processes, when the initial body A (or B) ceases, and (2) in the mid-point of the process, when the simpler forms, so called "kernel" and "hull" of A and B are created.

Table 2. Topological changes during truncation and epicuration process

(1) Changes at the moment of leaving the initial form	
To the truncated form	To the epicurated form
Old vertices split up, new faces are created "within" them They are separated from old faces by new edges, which intersect with old edges in new vertices All vertices are of three-fold type	Old faces split up, new vertices are created within them They are connected to old vertices by new edges, which, together with old edges, bound new faces All faces are triangular
(2) Changes in the mid-point of the process, at the moment of leaving	
The truncated form to the kernel	The epicurated form to the hull
Vertices meet each other in pairs at the same place All old edges disappear All vertices become four-fold	Faces melt in pairs All old edges disappear All faces become tetragonal

Transformation Circles Bringing epicuration and truncation together, duality appears as opposites in a circle—not only between the initial bodies A and B. All opposite bodies in the transformation circle are dual partners: truncated A is dual to epicurated B, kernel is dual to hull.

Kernel and hull are clearly assigned to one dual pair; while an outer transition form can be part of several transition circles. (Compare Fig. 14: Transformation Circles of Polyliner: There are equal epicuration forms in the first and in the third duoliner-pair, as well as in the third and the fifth triliner-pair.)

2 Further Structures Corresponding to Dual Pairs

The four-foldness of elements in the kernel and in the hull allows for further transformation.

2.1 Folding Down the Hull into Bundles

Constructive Reality When dealing with physical models, questions about stability and flexibility arise. The jump from the face's three-foldness to four-foldness causes a fundamental change in the constructive properties of the model. Made of flexible framework, all phases of epicuration are stable structures—it is only models of the hulls (in the mid-point of the process) that are flexible. They can be collapsed and routed to two different end positions in a regular way (Fig. 5).

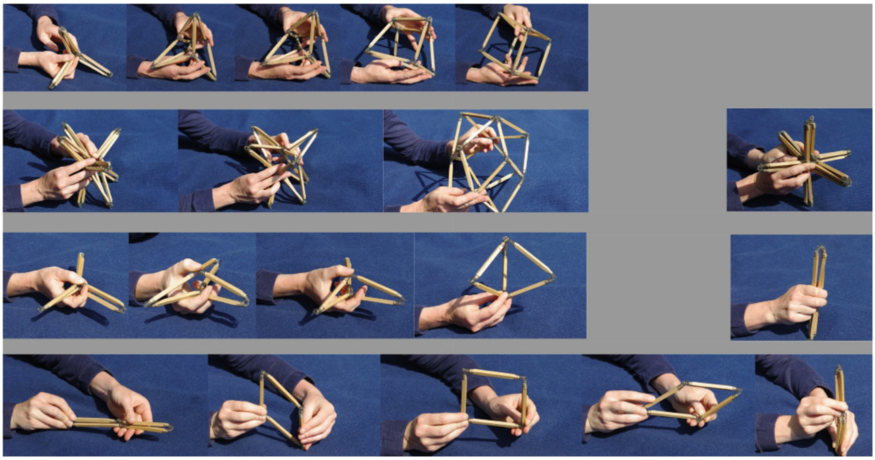

Fig. 5. Top: folding down the hull. Hull in centre, bundles left and right; 1st row: cube—the hull of tetrahedron and tetrahedron; 2nd row: rhombic dodecahedron—the hull of cube and octahedron; third row: hull of "triangular dipole and triangular cushion"; 4th row: hull of "duogonal dipole and dougonal cushion" (for row 3, 4 compare paragraph 3.1: polyliner with duogon)

Bundle as Connection of Centre Point and Peripheral Points These structures, realizing a rhythm between inside and outside, we call "bundles" of the dual pair. One of the bundles connects the centre point with all former vertices of A, the other one connects the centre point with all former vertices of B by a number of lines corresponding to the vertex's order.

Bundles result from continuing the intersection process as described in paragraph 1.2:

Changing gradually the relative size of A and B-after A's vertices lay in B's surface (Fig. 6, row 3 and 7), further shrinking of A does not change the shape of the shared volume. But the "epicuration-form" (blue)—as a connection of nearest vertices of A and of B—is again transformed, now situated within the volume of B, possibly turning into a non-convex body (Fig. 6, left column, row 2 and 8). During further shrinking this is topologically the same body as the convex epicuration form. When eventually A disappears in a point, the "epicuration form" turns into a bundle (Fig. 6, left column, row 1 and 9) (Table 3).

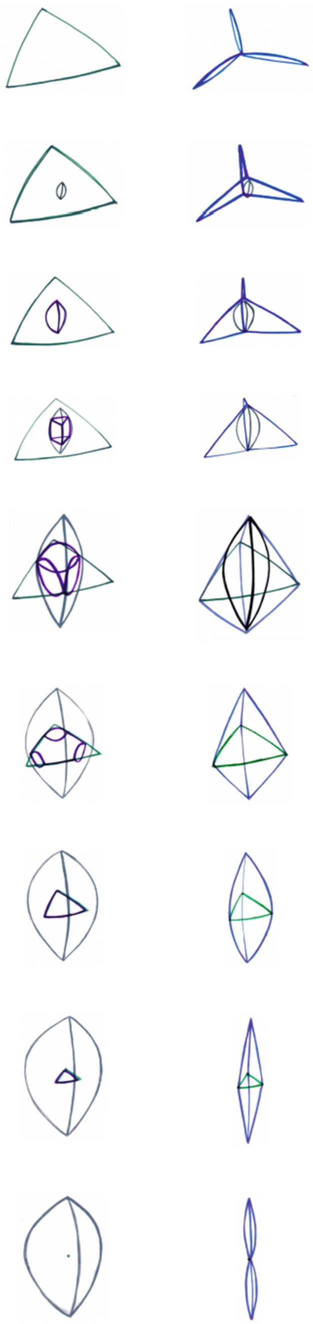

Fig. 6. Left: changing relative size of A and B (example: trigonal cushion and dipole, compare paragraph 3.1): left: both initial bodies A, B and kernel, right: the larger initial body and hull

Table 3. Development during further transformation from kernel to knot/link and from hull to bundle

Changes in the transformation from kernel to knot	Changes in the transformation from hull to bundle
In each of all four-fold vertices "opposite" aspects are identified The vertex splits up and two degenerated "two-fold" vertices are created	In each of all tetragons opposite points are identified The facet splits up and two degenerated "duogonal" facets are created

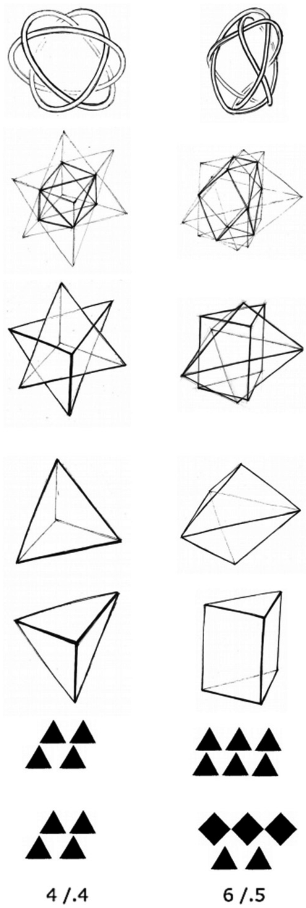

Fig. 7. Translation of dual pairs of polyhedra into alternating knots/links. From bottom to top: number of faces/vertices; face types of the polyhedron A; face types of the dual polyhedron B; polyhedron A (left: tetrahedron, right: trigonal prism); polyhedron B (left: tetrahedron, right: trigonal dipyramid); intersection A and B; intersection and core form; corresponding knot/link (left: the Borromean rings)

2.2 From the Kernel to Knots and Links

Again, the step from kernel to knot has an effect on the constructive properties of a model: It allows for an elegant and simple building method. From a polyhedron whose vertices are all four-fold (relating to a medial graph—compare paragraph 3.3) an edge model can be easily build by guiding a wire without cutting it (Figs. 7 and 8).

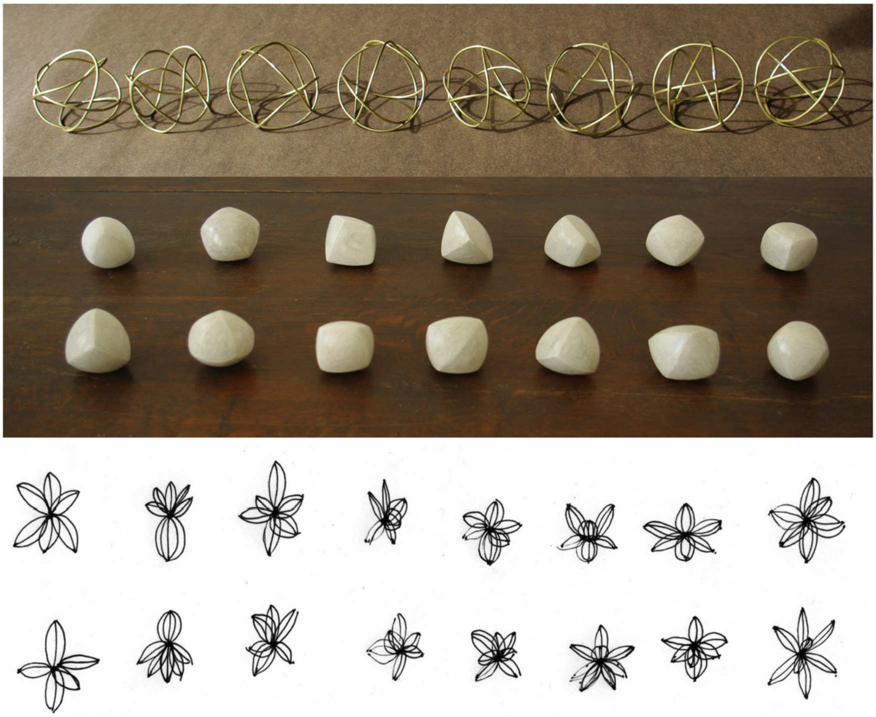

Fig. 8. Hexahedra, their knots/links and bundles (from left to right in the same order as in Fig. 2). Top: brass models with both chiral variants; middle: clay models with one chiral variant; bottom: drawings of the bundles

All surface disappears. Edges are interpreted as strands. Vertices split and gain a new function as crossing points, where the strand passes, altering once outside, once inside, creating an alternating knot or link.

Again a rhythm between inside and outside is opened up by the former edges: here swinging, as the track of a moving point in the knot, there flashing rapidly in the bundle between expansion and contraction.

The two knots are much more similar in appearance than the two bundles: they differ only in their handedness. (A self-dual polyhedron includes an amphichiral knot). Furthermore, a knot is clearly assigned to a dual pair, while a single bundle can correspond to several pairs (compare Fig. 13 column 5/6 and 7/8). Therefore, the knot may count as the central form of synthesis within a dual pair.

3 Simplest Cases of Dual Pairs, Knots, Links and Bundles

Simplest Knots The simplest pair of polyhedra—the tetrahedron and its dual tetrahedron—corresponds to the "Borromean Rings", a link of three strands. The Borromean rings form 6 crossing points corresponding to the 6 edges of the Tetrahedron (Fig. 7). But that is not the simplest alternating knot! The tables show: There are knots and links with two, three, four and five crossing points (Fig. 9).

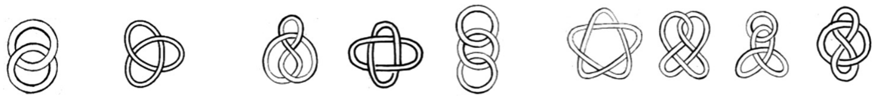

Fig. 9. Simplest knots—knots and links with 2–5 crossing-points

The incompleteness of the series of alternating knots and links, as they emerge from the synthesis of polyhedral pairs, suggests applying the principle of duality and the process of transformation to bodies with possibly less than four faces (Fig. 10).

Fig. 10. Generalized polygons. From left to right: monogon; duogon; trigon; tetragon; pentagon; hexagon

Polyliner Polyliner are bodies, whose surface is structured by facets, edges and vertices just as polyhedra. But polyliner's elements are generalized: faces and edges may be curved. In the following curved faces are referred to as facets, while we stay with the terms "vertex" and "edge". Concerning the principle of duality and the processes of transformation, polyliner can be treated as generalized polyhedra.

3.1 Polyliner with Duogon

Curvature allows for introducing duogonal facets and from these compounding bodies with less than four facets. The corresponding element to a duogonal facet is a twofold vertex, wherefrom only two edges part and where-in only two facets meet. By introducing duogons, a corresponding dual pair of polyliner is found even for the simplest knots.

Examples: Duo- and Triliner The first newly created body with duogonal facets is a "beechnut" shape, enclosing volume between three facets, which meet in two threefold vertices (Fig. 11, right column, 5th row). Its dual partner spans two trigons between

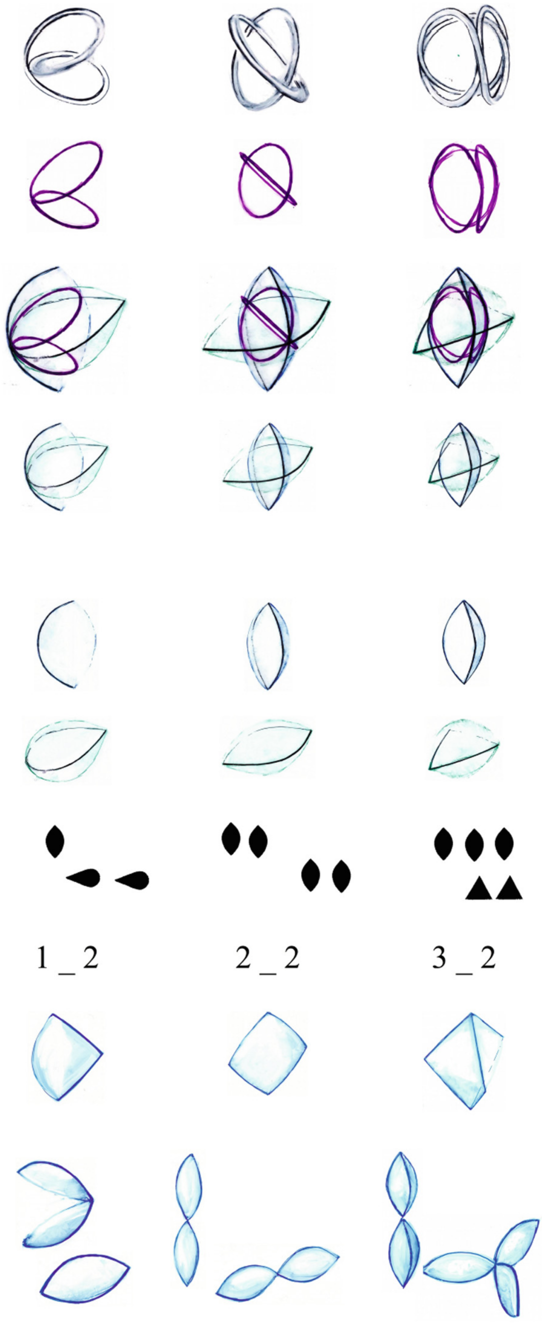

Fig. 11. Mono- duo- and triliner. From bottom to top: bundles; shell form; intersection A; B and shell form; number of facets A and B; facet types B; facet types A; polyliner B (monogonal, duogonal and trigonal cushion); polyliner A (one-fold, two-fold and three-fold dipole); intersection A and B; intersection and kernel; kernel; corresponding knot/link/spatial line with loop (centre: Hopf-Link, right trefoil knot)—(for left column see paragraph 3.2: polyliner with monogon)

three two-fold vertices, like a "triangonal cushion". Again, the transformation process delivers a corresponding knot, known as the "Threefoil Knot".

One step further in reduction, two duogons meeting in two two-fold vertices resemble a "pod"-like, self-dual body corresponding to the "Hopf-Link". With these two dual pairs, the knot and link with two and three crossing points are found.

Groups of Finite and Infinite Order—Counting Edges There is an infinite number of polyliner, either unfolding n duogons between 2 vertices ("dipoles") or composing 2 facets of n-gonal, cushion-like curved shape ("cushions"). Polyliner allow themselves to be classified into groups of finite order, where the number of edges is decisive. Therefore we suggest the term "polyliner" and sort the following templates by the number of polyliner's edges.

Tight Bending in Tetraliner and Pentaliner—Unknotting without Monogon Curvature of edges and facets was the key for the creation of polyliner. Extreme curvature allows a polygon to encircle the body's volume by identifying some of its points or edges with each other. If a tetragon is bent so tightly that two opposite vertices meet, then a tetraliner can be compounded, which gathers two dougons with the tetragon and is self-dual (Fig. 12, 3rd column). For delivering polyliner-pairs for all the variants in the knot table, we need tightly bent facets on board.

In the group of pentaliner tight bending happens—point to point in a pentagon (Fig. 12, 7th column)—and further, there is a hexagon with two of its edges identified (8th column). (In general edges of two facets are identified with each other, here it is edges of one and the same facet.) Corresponding is a speciality in the link: by twisting its two halves, the centre crossing point can be untangled: the link partly unknots.

3.2 Polyliner with Monogon

The newly acquired freedom to allow curved surfaces and curved edges as structuring elements suggests that even polyliner with "monogons"—facets with one point—may conform with the principle of duality. Monogons are bounded by only one edge, which returns to itself at a single point (Fig. 10). The corresponding situation in the dual partner is a vertex, wherefrom only one edge emerges and wherein a single facet meets itself—a so called one-fold vertex.

Examples: Monoliner The simplest body with monogon has a single edge, two monogonal facets and a two-fold vertex, resembling a "seed"-like form. It is a member of the cushion family. Its dual—the most simple dipole—has two one-fold vertices and one duogonal facet. In its single edge the two edges of its tightly bent duogon come together, resembling a "purse"-like form (Fig. 11, left column).

Regular Polyliner In a regular polyhedron all faces are equal in type and all vertices are equal in type. All cushions and dipoles conform with this restriction, but only the one- two- and three-fold variants can also—like regular polyhedra—result from an even distribution of points on the spherical surface, and therefore may count as regular polyliner.

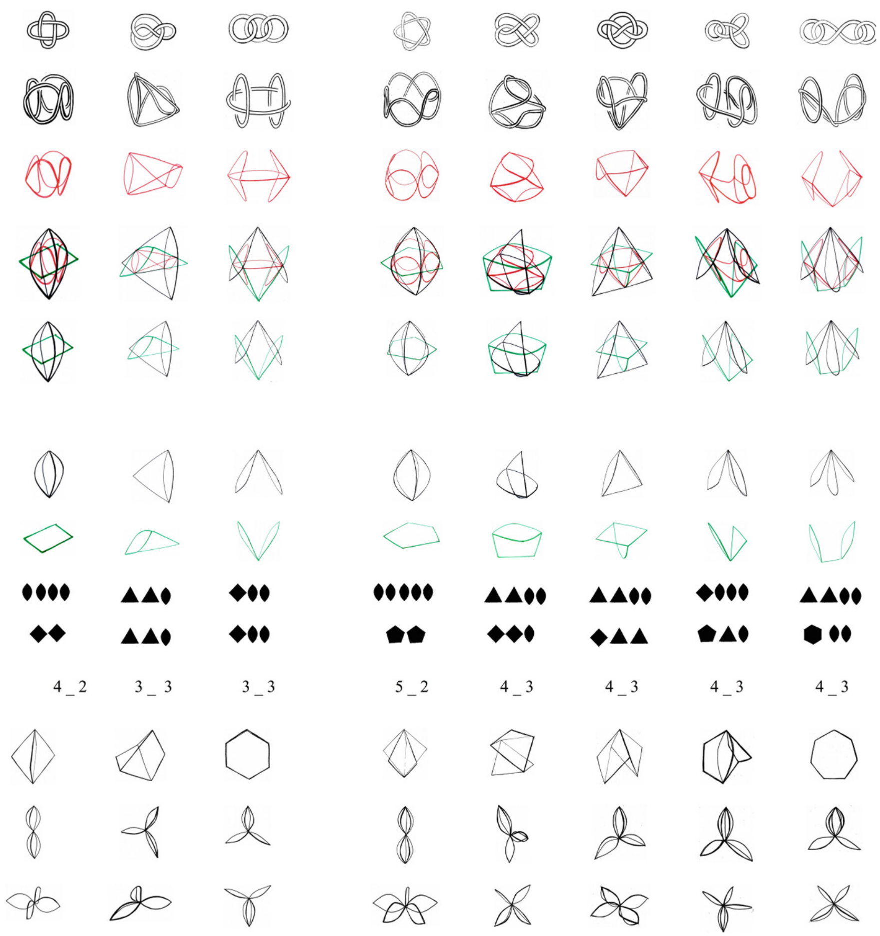

Fig. 12. Tetra- and pentaliner—polyliner with duogon, without monogon. From bottom to top: bundles; hull; number of facets A and B; facet types B; facet types A; polyliner B; polyliner A

3.3 Loops—Unknotting with Monogon

The situations of one-fold vertices and monogons correspond to a situation in a knot, where the strand is loosely twisted over itself. This apparent crossing, which can be easily undone (by Reidemeister Movement 1), is referred to as "loops". In the simplest case—the described monoliner-pair—there is only one such loop at all, and in terms of relative positional relationship such a topological structure would initially be regarded

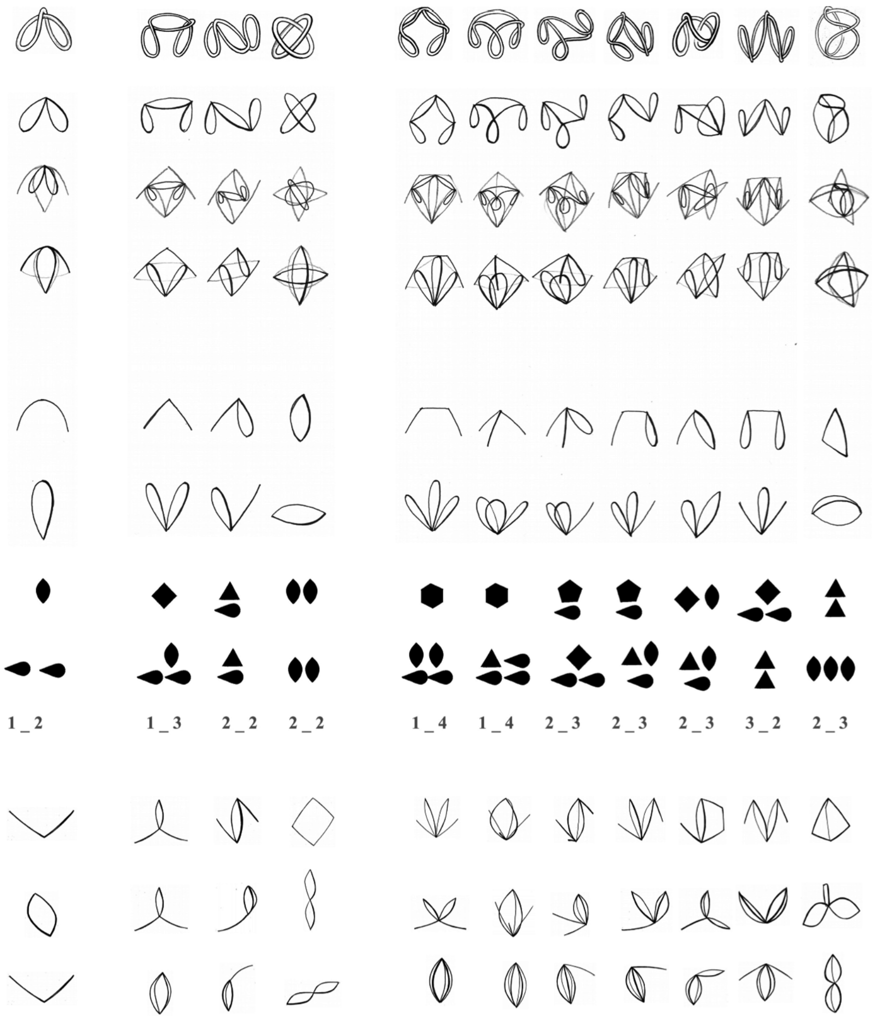

Fig. 13. Polyliner with monogon and duogon—monoliner, duoliner and triliner. From bottom to top: bundles; hull; number of facets A, B; facet types of polyliner B; facet types of its dual A; body B; body A; intersection A and B; intersection and kernel; corresponding knot/link/spatial line with loop

as a simple, unknotted ring. The spatial curve of the pair of triliner in Fig. 13, column 9, partly unknots into the Hopf-Link. However, if the loops are considered to persist with the strand's crossing point, then, due to the dual pair assignments, spatial line with loops can be catalogued and groups of finite order established (Figs. 13 and 14).

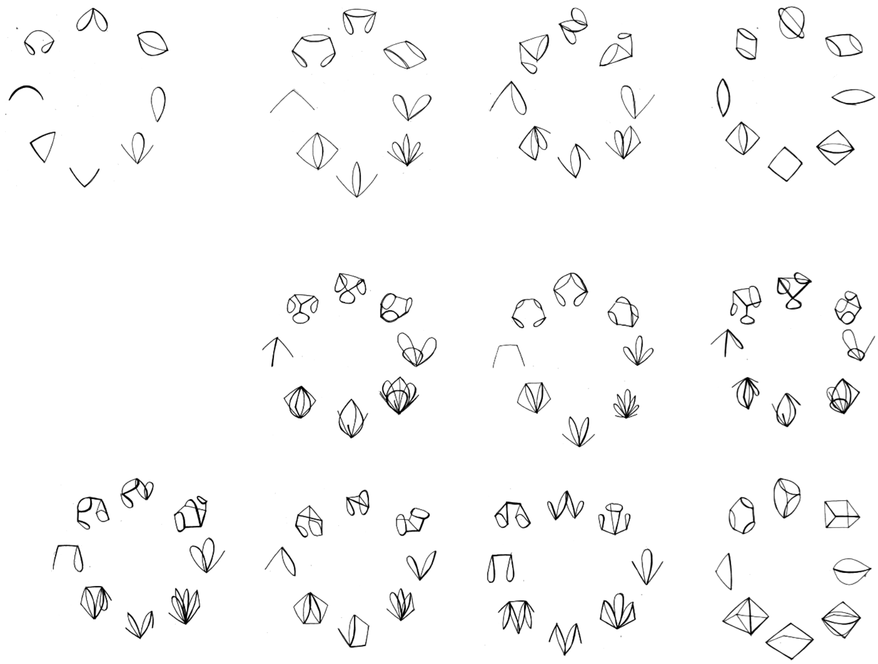

Fig. 14. Transformation circles of polyliner. 1st row: monoliner and duoliner; 2nd and 3rd row: triliner. Each clockwise: A; truncated A; kernel; truncated B; B; epicurated B; hull, epicurated A

3.4 Graphic Representation

In graph theory, the dual graph *B* of a plane graph *A* is a graph that has a vertex for each face of *A*. The outside of the graph counts as a face too. (This is the facet, through which the spatial net is projected.) The dual graph has an edge whenever two faces of *A* are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge of *A* has a corresponding dual edge in *B* and both have a crossing point.

The medial graph (corresponding to the kernel) is usually created from one of the initial graphs as another graph that represents the adjacencies between edges in the faces of *A*. Figure 15 results from a more harmonious viewpoint, starting from both graphs *A* and *B*—the dual pair. To gain the medial graph, all crossing points from the interlaced *A* and *B* are connected within each face.

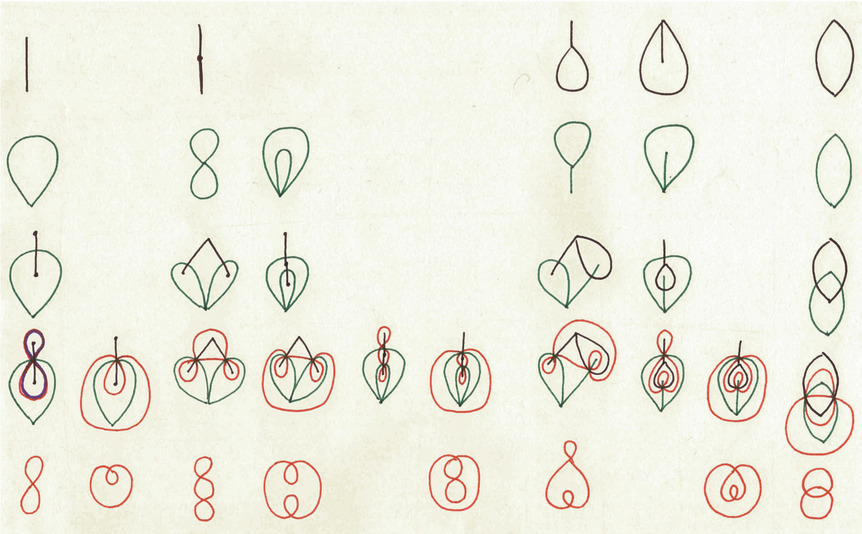

Fig. 15. Planar representation of the mono- and duoliner—all possible projections. From top to bottom: graph A; dual graph B; interlaced A and B; A, B and medial graph; medial graph