Duality in Non-polyhedral Bodies Part II
Triplets of the Polyliner

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Abstract. In a triplet group, there is a close relationship between three bodies, which in some respects is comparable to the relationship between two bodies in a dual pair. In both groups - dual pair and triplet - the bodies are completely equivalent to each other. Both or all three can serve as the starting point for the formation of the partner(s), whereby the same formation rule is applied. The phenomenon of triplets leads to connections between shapes and groups of shapes in a new, precise sense. The discovery of triplets is based on the introduction of polyliner, which result on the one hand from interpreting spherical graphs as solid bodies, and on the other hand from a generalization of polyhedra. Based on the application of the Conway transformations of polyhedron geometry to polyliner, the article introduces *triplets*: a group of three solids whose truncation and pyramid (Conway kis operator) forms are topologically identical. Furthermore, the edge set of the common truncation, resp., pyramid form is the disjoint union of the edge sets of the triplet.

Keywords: polyliner, spherical graph, Conway transformation, kis, core, dual, ambo, needle, zip, truncation, triplet

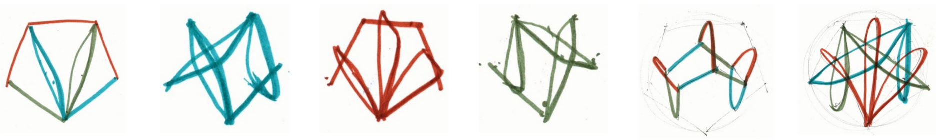

Fig. 1. Corresponding forms of a polyliner triplet.

Polyliner

In the article "Duality in non-polyhedral solids Part I: Polyliner" (ICGG Milano 2018), a new group of geometric solids was introduced. Polyliner arise from the generalization of polyhedra by allowing curved edges and faces as well as degree-2 vertices, order-2 faces (bigons), degree-1 vertices and order-1 faces (monogons) to structure their surface. For the structuring components of the polyliner, which could also be called "facettes, peaks and ridges", due to their possible curvature, the designations "face, vertex and edge" have been retained.

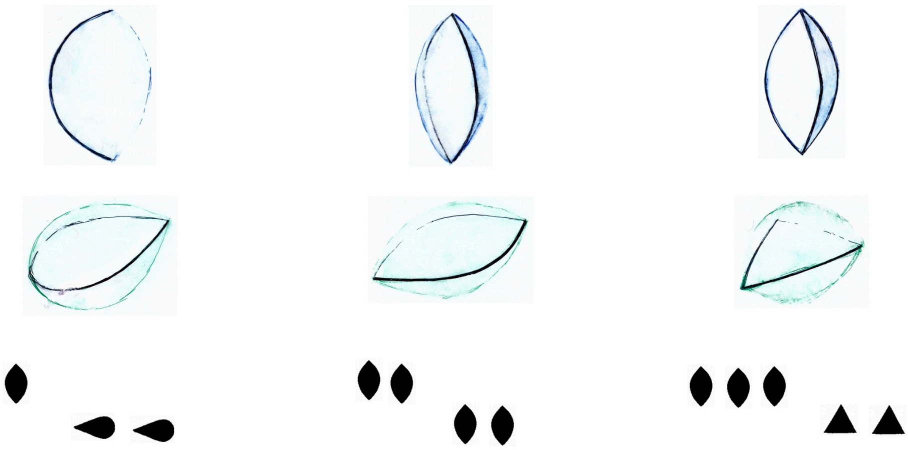

Fig. 2. Three dual pairs of geometric solids with curved edges and convex surfaces. From top to bottom: polyliner A; polyliner B (dual); notation for the generalized polygons shown. From left to right: monoliner, duoliner, triliner.

Representation of the polyliner - shorthand and notation. In the drawings in this article, polyliner are shown in a greatly reduced form. The lines of the shorthand used depict only the edges of the bodies. Contours of the bodies, where they are part of curved surfaces and not edges of the body, are not shown, in order to avoid misinterpretation of lines. They must be added mentally to enclose the volume.

Polyliner figures can be difficult to interpret for this reason, since parts of an edge may lie on the front surface of the polyliner and the rest may lie on the back surface. For this reason, edges on the back are drawn more delicately, those on the front more strongly. In order to reproduce the spatial orientation of the faces in a legible manner, the monogons in particular are often represented in a slightly distorted form.

The curvature of their components gives the polyliner an extended range of freedom for their form - in addition to measured edge lengths and face angles, the sharpness and path of their curving can be chosen. The topological type of the solid remains determined by the type and number of its faces, vertices and edges, as well as their relative position to each other.

Below each drawing appears notation for the polygons contained in it, in the interests of clear identification of the polyliner in question.

The transformation of polyliners into their pyramid form

As with every polyhedron, the principle of duality also assigns a dual partner to every polyliner, in which the roles of vertices and faces are reversed. Furthermore, the Conway transformations of polyhedron geometry can also be applied to polyliner. In particular, this article uses the operation ‘kis’ – pyramid formation on the faces, which we now describe.

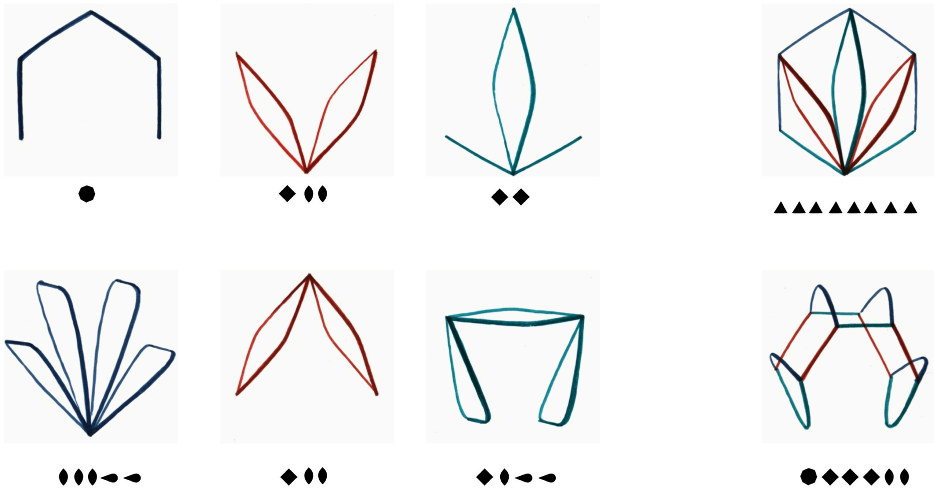

Fig. 3. Corresponding triplets. On the left the three contents, on the right their carrier, in the second row the duals of the first row; the middle of the three contents is self-dual. Below: truncation triplet; above: pyramid triplet. The first three polyliner on the top row have the same pyramid shape.

To create the pyramid shape, each face of an "original polyliner" is replaced by a pyramid. A new vertex is inserted above each face of the polyliner and connected to each vertex of the face on which it is based. If the face of the original polyliner has four vertices, a pyramid is formed from four triangles. If the face of the original polyliner is order-2, a pyramid is formed from two (curved) triangles. The newly inserted vertex therefore always has the degree of the order of the face of the original polyliner on which it is based.

The pyramid shape is a polyliner with new, exclusively triangular faces. It does not contain any faces from the original polyliner. In addition to the additional "new" edges of the pyramids, the "old" edges of the original polyliner are also included. This means that the number of edges of the pyramid shape is three times that of its original polyliner. The vertices of the original polyliner are also contained in the pyramid shape, but each now has doubled degree. The "old vertices" are therefore always of even degree. The newly inserted vertices, which form the tips of the pyramids, have the degree of the faces on which they are based.

Decomposition of the pyramid shapes of polyliner with even-order faces

If all faces of the original polyliner are even-order, all new vertices in the pyramid shape are also even-degree. In these cases, it is possible to swap the roles of the old and new vertices in the pyramid shape. To do this, starting at any vertex, every second edge meeting at this vertex is chosen. Proceed in the same way at each vertex encountered in this process. A 2-fold vertex forms a dead end and an end point of the edge subnetwork. In this way, a continuous set of edges is extracted; some vertices of the pyramid shape are left out.

Re-starting the process at a remaining edge leads to a second edge set, leaving out other vertices. In total, the edges of the pyramid shape are divided into three sets. To each edge set we associate the group of vertices not included in its edges. This makes it possible to complete each of the edge sets to a genuine polyliner. We complete the edge set by adding a face corresponding to each of the vertices left out of the given edge set. The position and numbering of these extra vertices indicates the type and position of the faces to be introduced.

Triplets of the Polyliner

The resulting group of three polyliner is called a "triplet". Note that each of the three can serve as a starting point for forming the same pyramid shape. Since the three polyliner, at least according to their vertices and edges, are contained in the pyramid shape, we call each one a "content of the triplet" and their common pyramid shape the "carrier of the triplet".

The phenomenon of triplets is apparently not described in the literature. It seems that the introduction of the polyliner was the necessary basis for its discovery.

From the point of view of polyhedron geometry, from which the polyliner appear as a generalization, it is not possible to find triplets: Although polyhedra can be the carrier and content of triplets, no triplet can consist exclusively of polyhedra (see section: Relations among vertex and face counts of triplet contents).

From the perspective of graph theory, the formation of dual and medial graphs from spherical graphs is familiar. (In their solid interpretation as polyliner, these correspond to the dual partner (Conway's "dual") and the core form (Conway's "ambo")). However, those spherical graphs which represent the pyramid forms - as well as their dual partners, the truncation forms - do not yet seem to have received any attention in their triplet-forming properties.

Emergence of truncation forms from the interlacing of dual polyliner

In polyhedron geometry, the Conway operations "ambo" and "truncation" can be defined as the cutting off or truncation of vertices. In polyliner geometry, the curvature of the newly created faces must also be taken into account so that all resulting core shapes and truncation shapes are solids enclosing volumes. Here it is appropriate to assume the interlacing of both initial bodies, so that the cutting/truncating tool on a polyliner is its own dual partner.

Different relative sizes of the two dual partners lead to three different forms.

Core and hull. If both solids are aligned in such a way that edges touch each other in pairs, the core shape of A and B (Conway's "ambo") is formed by intersecting their surfaces, yielding the intersection of their volumes (fig.4, center column). The "hull of A and B" (Conway's "join") is formed using the same relative sizes by connecting each vertex of one body with the vertices of the dual face.

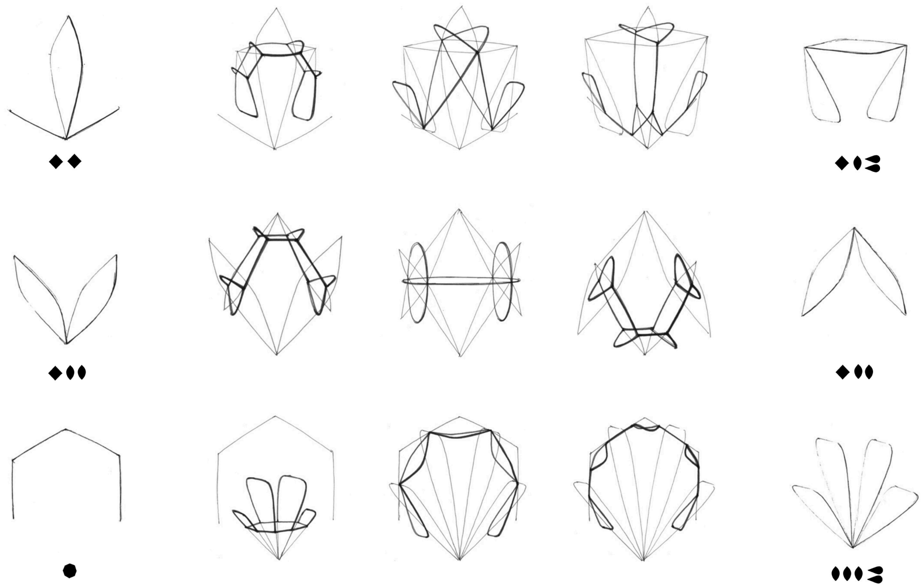

Fig. 4. Interweaving of dual partners in three size ratios. On each row, from left to right: polyliner A; interweaving of A larger than B with truncation form of B; interweaving of A and B in balanced size ratios with core of A and B; interweaving of A smaller than B with truncation form of A; polyliner B. From top to bottom: 3 dual pairs of tetraliner. Note: the truncation forms of B are topologically identical (second column).

All edges of the shapes core and hull are new edges, created as intersecting lines between faces or as connecting lines between vertices. This means that the core and hull each have twice as many edges as their original polyliner. Core and hull shapes are always clearly associated with a dual pair of original shapes.

Truncation shapes and pyramid shapes. If the size ratios of the two original solids are changed, the edges lose their contact points and, if A is smaller than B, the truncation shape of polyliner A (Conway's "zip") is formed as the intersection of the volume and, if B is smaller than A, the truncation shape of polyliner B (Conway's "needle") is formed. Truncation shapes combine edges that are newly created as intersecting lines between the faces of A and B with pieces of edges that were already part of the edges of an original body. This means that the number of edges of the truncation shapes is three times that of their original bodies.

Corresponding triplets

According to the principle of duality, triplets always occur in pairs. The dual partners of the contents of the first triplet are the contents of the second (dual) triplet. And in order to obtain the carrier of the second triplet, truncation – the dual operation to forming the pyramid form – is applied to the dual contents.

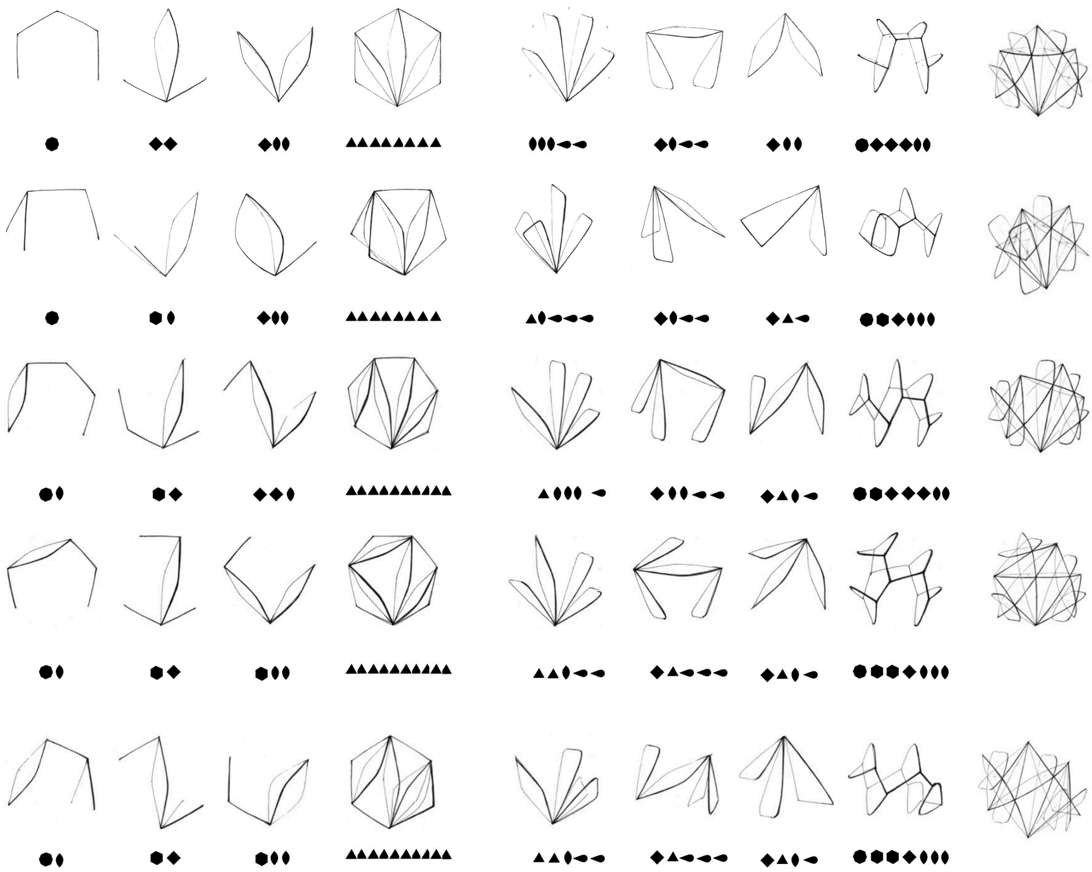

Fig. 5. Triplets type a,b,c. From left to right: contents of the pyramid triplet (polyliner a; b; c); carrier of the pyramid triplet; contents of the truncation triplet (dual partner of a; b; c); truncation triplet carrier; interlacing of the three truncation triplet contents (in the right hand margin). From top to bottom: triplets of tetraliner, pentaliner.

Prerequisites for the occurrence of triplets. The occurrence of triplets depends on certain properties of the original polyliner: Polyliner that consist exclusively of even-order faces are always the content of a pyramid triplet, and polyliner that contain only even-degree vertices are always the content of a truncation triplet.

Relationships between content and carrier in pyramid triplets and truncation triplets. The way in which the two groups of three meet - the content of the pyramid triplet versus the content of the truncation triplet - is different in many respects. The two starting positions, consisting of dualized original forms and dualized operations to generate their triplet carrier, lead to a dualized structure of relations.

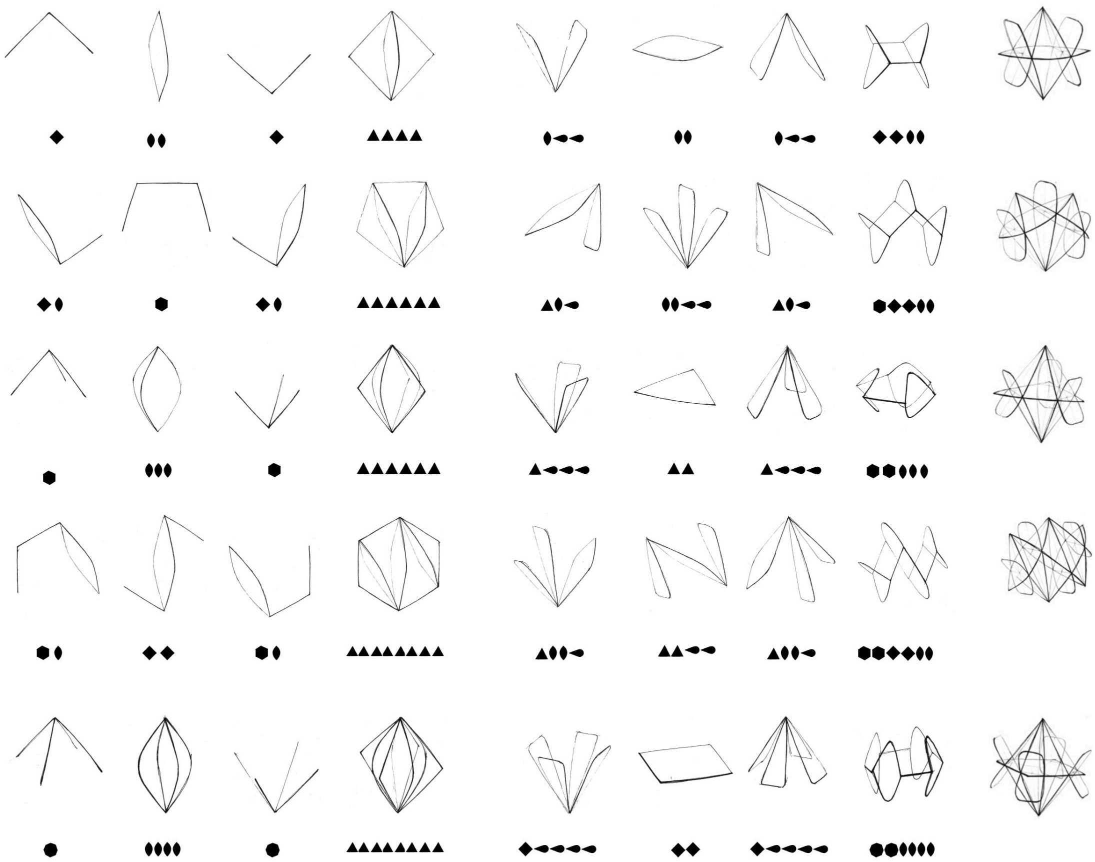

Fig. 6. Triplets type a,b,a. From top to bottom: triplets of duoliner, triliner, tetraliner. From left to right as in Fig. 5.

The carrier of the truncation triplet is obviously smaller than its contents, but the carrier of the pyramid triplet is larger than its contents. The edges of all three contents are contained in their full length in the pyramid triplet carrier, but a piece has been cut off in the truncation triplet carrier. This is why the pyramid triplet was initially easier to decipher than the truncation triplet.

The dual function of the edge. The apparent inequality of the two situations is deceptive, however. This is because edges have a dual function, simultaneously connecting vertices and separating faces.

In the pyramid triplet, the length of the edge - as a connection between two vertices - is identical in the carrier and the content, but the dihedral angle at this edge is flatter (that is, smaller) in the carrier than in the content. In the truncation triplet, the dihedral angle between the two faces adjacent to the edge is identical in the carrier and the content, but the length of the edge is shorter in the carrier than in the content.

Decomposition of a truncation triplet carrier into its three contents

The edges of the truncation triplet carrier can also be decomposed. To do this, every second of the edges surrounding it is extracted, starting at any face (fig.3, second row). The same procedure is carried out on each face to which the initially extracted edges are attached. In this way, a set of edges is obtained. The edges do not share any common vertices, but they do share common faces. However, some of the faces of the carrier are not reached. To complete the edge set for the polyliner, the vertices and the parts of the edges adjacent to them must be added. They lie outside the carrier, as is to be expected after they have been cut off during the truncation operation.

Three Types of Triplets

A systematic analysis of all polyliner with even-degree vertices and even-order faces shows that three types of triplets exist: "self-triadic triplets", which contain three identical polyliner (type a,a,a) (Fig. 8); triplets with three different contents (type a,b,c) (Fig.5), and triplets with two identical and one different content (type a,b,a) (Fig.6).

The rarest of these are the triplets type (a,a,a). Only polyliner with edge counts of the form (3n-2) - i.e. mono-, tetra-, heptaliner and so on. - can be the content of triplets of type (a,a,a). In addition, a certain ratio between the number and count of their vertices and faces must be maintained.

Relations among vertex and face counts of triplet contents. In the pyramid triplet, there are n-degree vertices of a partner content above the 2n-order faces of a content. In the truncation triplet, there are n-order faces of partner content under the 2n-degree vertices of a content. This means that contents of self-triadic pyramids combine triplets k 2n-order faces between 2k n-degree vertices, and contents of self-triadic truncation triplets combine 2k n-order faces between k 2n-degree vertices.

The relation among the counts of the vertices and faces of the contents of a triplet also prevents the creation of triplets whose contents are all polyhedra. Polyhedra can only be part of triplets in combination with polyliner contents.

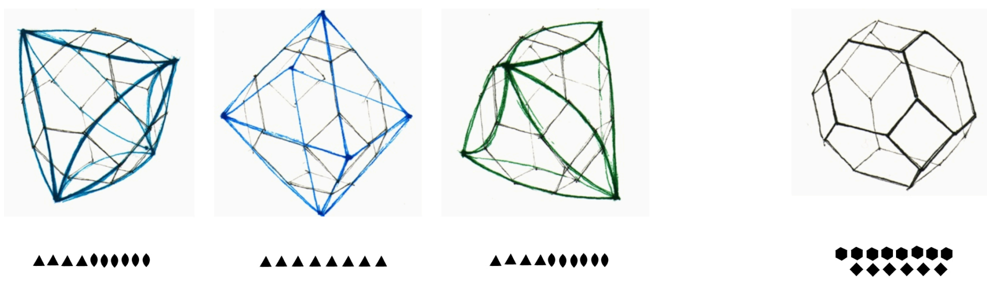

Fig. 7. Dodecaliner and octahedron as contents of a truncation triplet. The carrier is the truncated octahedron (right).

Octahedron and cube also appear as triplet carriers - see tetraliner triplet Fig. 8.

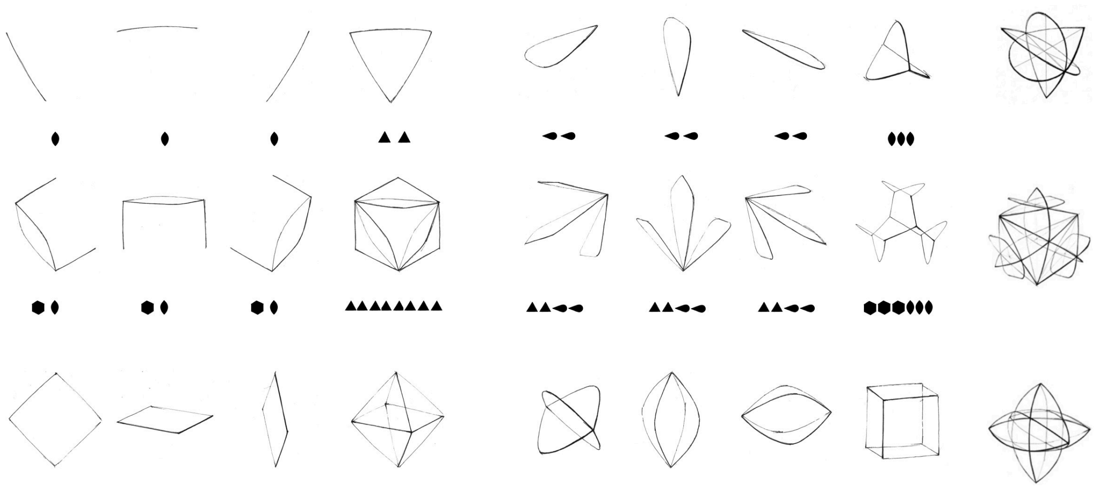

Fig. 8. Triplets type a,a,a. From top to bottom: Triplets of monoliner, tetraliner. From left to right as in Fig. 5.

Forms of interlacing the three contents

If the missing parts of the original solids are added to the triplet carrier - in the case of the truncation triplet, vertices and their adjacent edge length outside the carrier, in the case of the pyramid triplet, faces and their adjacent dihedral angles inside the carrier - two types of interweaving figures are created. Both figures interweave three solids equally, but in different ways: In the pyramid triplet, the bodies of the contents are stitched together at vertices on the outside; in the structure of the truncation triplet, the faces of the original bodies coincide inside the interlocking figure and form the carrier as a stable core.

Triplet and dual pair

Summarized in this way, the equivalence of the three partners in the truncation triplet is particularly clear. Their mutual interlacing, the evenly distributed points of contact between the edges of the different contents, the alternation in the bulging out and receding of volume components are reminiscent in their appearance of the interlacing of two bodies of a dual pair. The similarities and differences between dual pairs and triplets are summarized below.

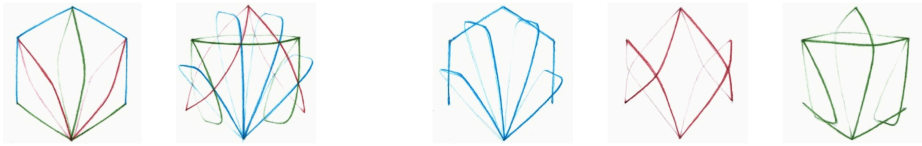

Fig. 9. Interweaving figures of the 6 corresponding contents. From left to right: group of three of the pyramid triplet; group of three of the truncation triplet; the same 6 solids in the three interweaving figures of the dual pairs.

Similarities between triplet and dual pair

- The partners play equivalent roles in every respect.
- Each partner in a group has the same number of edges.
- Joint interlocking is possible for the two bodies of the dual pair as well as for the three bodies of the truncation triplet, so that each edge of a body touches an edge of the partner(s). In the interlocking figures, vertices and adjacent volume sections protrude in alternating star shapes.
- Interlocking is the basis for the formation of core and hull shapes.

Differences between triplet and dual pair

- A dual pair consists of two equivalent bodies, while the triplet consists of three.
- Each polyliner has a dual partner. - Triplets arise only from polyliner that fulfill certain requirements.
- In the intersection of the dual pair, edges meet in pairs at one vertex. There is a contact point on each edge. - In the interweaving of the truncation triplet, three edges meet at one point. There are two contact points on each edge.
- A polyliner can be dual to itself (self-dual). Self-duality can occur in polyliner with an even number of edges. - There are three different types of triplets, depending on the repetition of their contents: those in which all partners are the same; those in which all partners are different; and those in which there is a third next to two identical ones. Polyliner with a 3n-2 number of edges can be content of a self-triadic triplet.

Cores and hulls of the triplet interlocks

The interlacing of three polyliner in the truncation triplet and the similarity of their appearance to the dual pair suggest the formation of core and hull forms of the interlocking of three polyliner. In fact, the carrier of the truncation triplet already contains its core, and the carrier of the pyramid triplet already contains its hull.

The hull of the truncation triplet can be easily formed by connecting the vertices of the three contents. The resulting shape corresponds to the dual partner of the core, i.e. the carrier of the corresponding pyramid triplet. Less obvious is the formation of the core in the intersection of the three pyramid triplet contents.

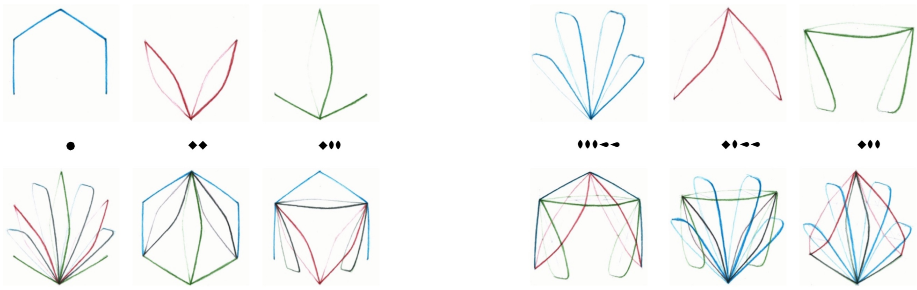

Fig. 10. 2 /3 Triplets. Top left: three contents of the pyramid triplet; below: each two of the pyramid contents generate the dual partner of the third, missing content as the core shape of their intersection; top right: three contents of the truncation triplet; below: each two of the truncation contents generate the dual partner of the third, missing content as the hull shape of their intersection.

2/3 Triplets. In order to determine the intersecting lines of their faces in a clearer situation, it makes sense to first look at two of the three contents in isolation. Three pairs from each triplet are possible: (a,b), (b,c), (c,a). When looking at their core and hull forms, the first thing that stands out is the strong cohesion across the carrier: The identity of the carrier as the core form of the truncation triplet or as the hull form of the pyramid triplet is retained in every case, even if only two of the three contents are interlocked. However, the hull forms of two truncation triplet contents and the cores of two pyramid triplet contents change: Two contents of the truncation triplet form the dual partner of the third content as a hull - by joining their vertices. And two contents of a pyramid triplet form the dual partner of the third content as a core shape - in the common volume of their surfaces. As a consequence, in the interlacing of all three contents of the pyramid triplet there can be found an interlacing of the three truncation triplet contents and finally, as the core of the whole, the carrier of the truncation triplet. The interweaving of the three pyramid triplet contents and the interweaving of the three truncation triplet contents therefore have the same core and the same hull.

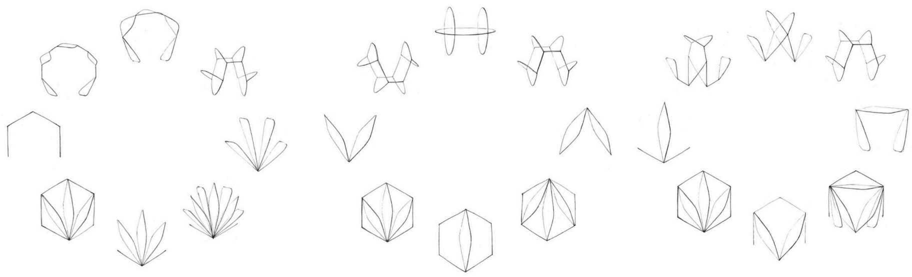

Fig. 11. Dual pair transformation circle. Three dual pairs of tetraliner in Conway transformation. Clockwise, starting left: polyliner A; truncation of A; core of A and B; truncation of B; polyliner B; pyramid form of B; hull of A and B; pyramid form of A.

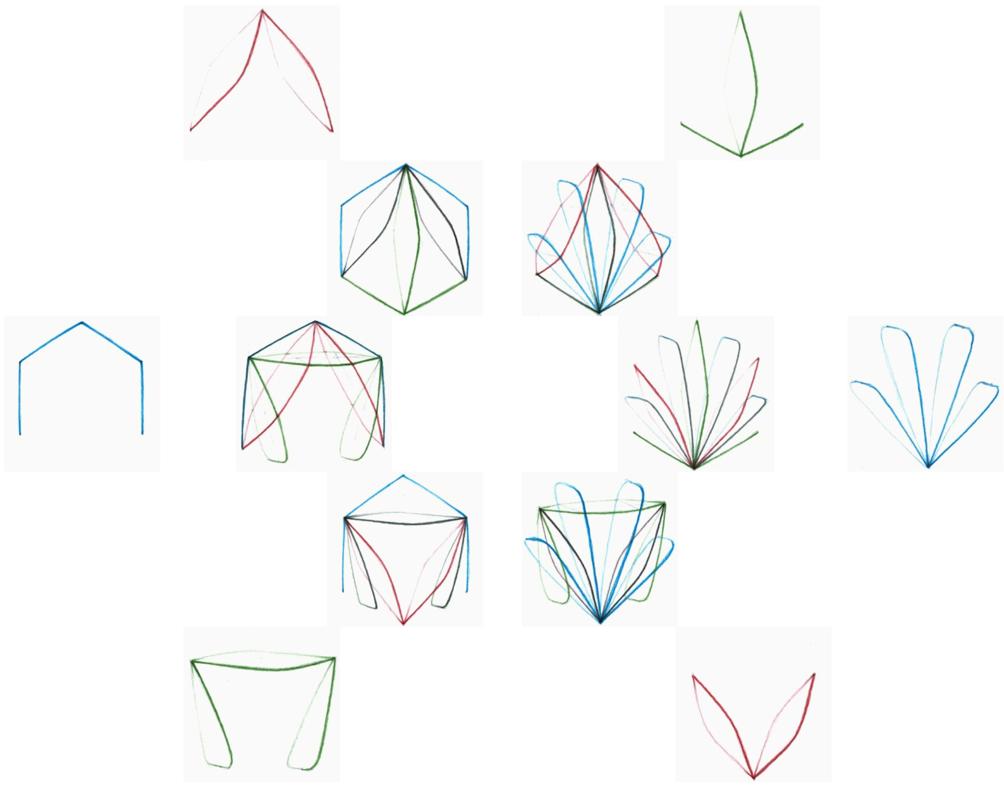

Fig. 12. Triplet circle. Outside: 6 corresponding tetraliner; inside: 2/3 interlacing of the two neighbors in the circle. The contents of the pyramid triplet (top right, bottom right, center left) generate the contents of the truncation triplet in their 2/3 intersections as a core. The contents of the truncation triplet (top left, bottom left, center right) generate the contents of the pyramid triplet as a hull via their 2/3 intersections.

Dual pair transformation circle and triplet circle

The circular arrangement of the eight bodies arising from Conway transformation. The arrangement of the eight solids in the "dual pair transformation circle" places dual bodies opposite each other (fig.11). The basic structure of the dual-pair transformation circle corresponds to two semicircles. On the horizontal dividing line are the original shapes, in the upper semicircle the core shapes of their interlacing with different size ratios, in the lower semicircle their hull shapes. This arrangement also illustrates the fact that continuous transformation processes between the individual bodies transform a polyliner into its dual partner - and back again.

The dual pair transformation circle contains three types of shapes that differ in their number of edges. Half of the shapes can also occur in other dual-pair transformation circles, provided they have only even-order faces and/or only even-degree vertices.

The circular arrangement of the six contents of corresponding triplets. The circular arrangement of the six original bodies in corresponding triplets is suggested by the differentiated relationship structure of the core and hull forms of 2/3 triplets.

In the "triplet circle" there are three dual pairs facing each other. All three pairs are of equal value. The basic structure of the arrangement corresponds to two interweaving triangles, the triangle of truncation-triplet contents and the triangle of pyramid-triplet contents. Each body can be obtained from the 2/3 interlocking of its two direct neighbors. Polyliner that consist exclusively of even-order faces and also have only even-degree vertices occur in two triplet circles (e.g. tetraliner in Fig. 6 and 8).

Conclusion

The structure of the six original forms in corresponding triplets is characterized by the equivalence of the bodies and spans a field of reciprocity. Similar to a dual pair, the interplay of the bodies in the triplet leads to the creation of synthetic forms, in particular cores and hulls. There is thus a triadic relationship of balance and complementation that reaches into the foundations of spatial structure.

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Graduate sculptor, master student of Inge Mahn. Artistic research on mathematical issues: Corpuscle geometry (units of flexible spatial parquetry), polyliner (three-dimensional representatives of numbers), inversion (the coherence of divided space in its elastic processes). Exhibitions of Concrete Art (Rappaz Museum Basel) and in Art&Science projects (Goethe Institute Nancy, Institute for Computer Science FU Berlin, Museum for Minerals and Mathematics Oberwolfach). Publications and lectures at international conferences (Rutgers University New Jersey, TU Kyoto, Polytechnic Milano). Teaching assignments and seminars on projective and experimental geometry (Muthesius Academy of Fine Arts Kiel, FHNW University of Art and Design Basel).

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